Evolutionary games on the lattice: payoffs affecting birth and death rates

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Abstract This article investigates an evolutionary game based on the framework of interacting particle systems. Each point of the square lattice is occupied by a player who is characterized by one of two possible strategies and is attributed a payoff based on her strategy, the strategy of her neighbors and a payoff matrix. Following the traditional approach of evolutionary game theory, this payoff is interpreted as a fitness: the dynamics of the system is derived by thinking of positive payoffs as birth rates and the absolute value of negative payoffs as death rates. The nonspatial mean-field approximation obtained under the assumption that the population is well mixing is the popular replicator equation. The main objective is to understand the consequences of the inclusion of local interactions by investigating and comparing the phase diagrams of the spatial and nonspatial models in the four dimensional space of the payoff matrices. Our results indicate that the inclusion of local interactions induces a reduction of the coexistence region of the replicator equation and the presence of a dominant strategy that wins even when starting at arbitrarily low density in the region where the replicator equation displays bistability. We also discuss the implications of these results in the parameter regions that correspond to the most popular games: the prisoner's dilemma, the stag hunt game, the hawk-dove game and the battle of the sexes.

1. Introduction

The book of von Neumann and Morgenstern [17] that develops mathematical methods to understand human behavior in strategic and economic decisions is the first foundational work in the field of game theory. The most popular games are symmetric two-person games whose characteristics are specified by a square matrix where the common number of rows and columns denotes the number of possible pure strategies and the coefficients represent the player's payoffs which depend on both her strategy and the strategy of her opponent. Game theory relies on the assumption that players are rational decision-makers. In particular, the main question in this field is: what is the best possible response against a player who tries to maximize her payoff? The work of Nash [14] on the existence of Nash equilibrium, a mathematical criterion for mutual consistency of players' strategies, is an important contribution that gives a partial answer to this question.

In contrast, the field of evolutionary game theory, which was proposed by theoretical biologist Maynard Smith and first appeared in his work with Price [13], does not assume that players make rational decisions: evolutionary game theory makes use of concepts from traditional game theory to describe the dynamics of populations by thinking of individuals as interacting players and their trait as a strategy, and by interpreting their payoff as a fitness or reproduction success. The analog of Nash equilibrium in evolutionary game theory is called ESS, a short for evolutionary stable strategy, and is defined as a strategy which, if adopted by a population, cannot be invaded by any alternative strategy starting at an infinitesimally small frequency. Even though evolutionary games were originally introduced to understand the outcome of animal conflicts, this concept now has a

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wide variety of applications as a powerful framework to study interacting populations in which the reproductive success of each individual is density dependent, an important characteristic of social and biological communities.

The inclusion of stochasticity and space in the form of local interactions is another key factor in how communities are shaped, and evolutionary games have been studied through both the mathematical analysis of deterministic nonspatial models based on differential equations and simulations of more complex models based on spatial stochastic processes. See [8, 16] for a review of these two aspects. Both approaches are important and complementary but also have some limitations: spatial simulations suggest that nonspatial models fail to appropriately describe systems including local interactions but are known at the same time to be difficult to interpret, leading sometimes to erroneous conclusions. This underlines the necessity of an analytical study of evolutionary games based on stochastic spatial models, which is the main focus of this paper. References [2, 4] are, as far as we know, the only two articles that also carry out a rigorous analysis of such models but their approach significantly differs from ours: they assume that

$$\text{fitness} \ = \ (1 - w) + w \times \text{payoff} \quad \text{and} \quad w \to 0$$

which is referred to as weak selection and allows for a complete analytical treatment using voter model perturbation techniques. Indeed, for $w = 0$, their model reduces to the popular voter model introduced in [3, 9]. In contrast, we assume that fitness = payoff, which makes our model mathematically more challenging and does not allow for a complete analysis. However, the limiting behavior in different parameter regions can be understood based on various techniques, which leads to interesting findings about the consequences of the inclusion of local interactions.

The replicator equation. As previously mentioned, most of the analytical works in evolutionary game theory are based on ordinary differential equations. The most popular model that falls into this category is the replicator equation, which we describe for simplicity in the presence of only two strategies since this is the case under consideration for the stochastic spatial model we introduce later. The dynamics depends on a 2×2 payoff matrix $A = (a_{ij})$ where a_{ij} denotes the payoff of a player who follows strategy i interacting with a player who follows strategy j. To formulate the replicator equation and describe its bifurcation diagram, it is convenient to use the terminology introduced by the author in [10] by setting

$$a_1 \ := \ a_{11} - a_{21} \qquad \text{and} \qquad a_2 \ := \ a_{22} - a_{12}$$

and declaring strategy i to be

- **altruistic** when $a_i < 0$, i.e., a player with strategy i confers a lower payoff to a player following the same strategy than to a player following the other strategy,
- **selfish** when $a_i > 0$, i.e., a player with strategy i confers a higher payoff to a player following the same strategy than to a player following the other strategy.

The replicator equation is a system of coupled differential equations for the frequency u_i of players following strategy i. The payoff of each type i player is given by

$$\phi_i \ = \ \phi_i(u_1, u_2) \ := \ a_{i1}\, u_1 + a_{i2}\, u_2 \quad \text{for} \quad i = 1, 2. \tag{1}$$

Interpreting this payoff as the growth rate of each type i player, using that the frequencies sum up to one, and recalling the definition of a_1 and a_2, one obtains the following ordinary differential

equation, the so-called replicator equation, for the frequency of type 1 players:

$$\begin{aligned} u_1'(t) &= u_1\, u_2\, (\phi_1 - \phi_2) \ = \ u_1\, u_2\, (a_{11}\, u_1 + a_{12}\, u_2 - a_{21}\, u_1 - a_{22}\, u_2) \\ &= u_1\, u_2\, (a_1\, u_1 - a_2\, u_2) \ = \ u_1\, (1 - u_1)((a_1 + a_2)\, u_1 - a_2). \end{aligned} \tag{2}$$

The system has three fixed points, namely

$$e_1 := 1 \quad \text{and} \quad e_2 := 0 \quad \text{and} \quad e_{12} := a_2\, (a_1 + a_2)^{-1}$$

and basic algebra shows that the limiting behavior only depends on the sign of a_1 and a_2 therefore on whether strategies are altruistic or selfish. More precisely, we find that

- when strategy 1 is selfish and strategy 2 altruistic, strategy 1 wins: $e_{12} \notin (0,1)$ and starting from any initial condition $u_1 \in (0,1)$, $u_1 \to e_1$.
- when strategy 1 is altruistic and strategy 2 selfish, strategy 2 wins: $e_{12} \notin (0,1)$ and starting from any initial condition $u_1 \in (0,1)$, $u_1 \to e_2$.
- when both strategies are altruistic, coexistence occurs: $e_{12} \in (0,1)$ is globally stable, i.e., starting from any initial condition $u_1 \in (0,1)$, $u_1 \to e_{12}$.
- when both strategies are selfish, the system is bistable: $e_{12} \in (0,1)$ is unstable and u_1 converges to either e_1 or e_2 depending on whether it is initially larger or smaller than e_{12}.

In terms of evolutionary stable strategy, this indicates that, for well mixed populations, a strategy is evolutionary stable if it is selfish but not if it is altruistic.

Spatial analog. To define a spatial analog of the replicator equation, we employ the framework of interacting particle systems by positioning the players on an infinite grid. Our spatial game is then described by a continuous-time Markov chain η_t whose state space maps the d-dimensional lattice into the set of strategies $\{1,2\}$, with $\eta_t(x)$ denoting the strategy at vertex x. Players being located on a geometrical structure, space can be included in the form of local interactions by assuming that the payoff of each player is computed based on the strategy of her neighbors. More precisely, we define the interaction neighborhood of vertex x as

$$N_x \ := \ \{y \in \mathbb{Z}^d : y \neq x \text{ and } \max_{i=1,2,\ldots,d} \ |y_i - x_i| \leq M\} \quad \text{for} \ \ x \in \mathbb{Z}^d$$

where M is referred to as the dispersal range. Letting $f_j(x,\eta)$ denote the fraction of type j players in the neighborhood of vertex x, the payoff of x is then defined as

$$\phi(x,\eta \,|\, \eta(x) = i) \ := \ a_{i1}\, f_1(x,\eta) + a_{i2}\, f_2(x,\eta) \quad \text{for} \ \ i = 1,2,$$

which can be viewed as the spatial analog of (1). The dynamics is again derived by interpreting the payoff as a fitness. More precisely, we think of a payoff as either a birth rate or a death rate depending on its sign: if the player at vertex x has a positive payoff then, at rate this payoff, one of her neighbors chosen uniformly at random adopts the strategy at x, while if she has a negative payoff then, at rate minus this payoff, she adopts the strategy of one of her neighbors again chosen uniformly at random. This is described formally by the Markov generator

$$\begin{aligned} Lf(\eta) \ = \ & N^{-1} \sum_x \sum_{y \in N_x} \phi(y,\eta)\ \mathbf{1}\{\phi(y,\eta) > 0\}\ \mathbf{1}\{\eta(x) \neq \eta(y)\}\ [f(\eta_x) - f(\eta)] \\ & - N^{-1} \sum_x \sum_{y \in N_x} \phi(x,\eta)\ \mathbf{1}\{\phi(x,\eta) < 0\}\ \mathbf{1}\{\eta(x) \neq \eta(y)\}\ [f(\eta_x) - f(\eta)] \end{aligned} \tag{3}$$

where configuration η_x is obtained from configuration η by changing the strategy at vertex x and leaving the strategy at the other vertices unchanged, and where the constant N is simply the common size of the interaction neighborhoods. Model (3) is inspired from the spatial version of Maynard Smith's evolutionary games introduced by Brown and Hansell [1]. Their model allows for any number of players per vertex and the dynamics includes three components: migration, death due to crowding and a game step. Our model only retains the game step. We also point out that the model obtained from the spatial game (3) by assuming that the population is well mixing, called the mean-field approximation, is precisely the replicator equation (2), therefore the consequences of the inclusion of space and stochasticity can indeed be understood through the comparison of both models which, as we show later, disagree in many ways.

Main results. For simplicity, we study the limiting behavior of the spatial game starting from a Bernoulli product measure in which the density of each strategy is positive and constant across space, though some of our results easily extend to more general initial distributions. From the point of view of the replicator equation, whether a strategy wins or both strategies coexist or the system is bistable is defined based on the value of the nontrivial fixed point e_{12} and the stability of this and the other two fixed points. For the spatial game, we say that

- strategy $i \in \{1, 2\}$ **survives** if
$$P\,(\eta_s(x) = i \text{ for some } s > t) \ = \ 1 \quad \text{for all} \quad (x, t) \in \mathbb{Z}^d \times \mathbb{R}_+$$
and **goes extinct** otherwise,
- a strategy **wins** if it survives whereas the other strategy goes extinct,
- both strategies **coexist** whenever
$$\liminf_{t \to \infty} \ P\,(\eta_t(x) \neq \eta_t(y)) \ > \ 0 \quad \text{for all} \quad x, y \in \mathbb{Z}^d$$
- the system **clusters** whenever
$$\lim_{t \to \infty} \ P\,(\eta_t(x) \neq \eta_t(y)) \ = \ 0 \quad \text{for all} \quad x, y \in \mathbb{Z}^d.$$

Numerical simulations of the process suggest that, in the presence of one selfish strategy and one altruistic strategy, the selfish strategy wins, just as in the replicator equation. In contrast, when both strategies are selfish, spatial and nonspatial models disagree. Our brief analysis of the replicator equation indicates that the system is bistable: both strategies are ESS. The transition curve for the spatial model is difficult to find based on simulations but simple heuristic arguments looking at the interface between two adjacent blocks of the two strategies suggest that the most selfish strategy, i.e., the one with the largest a_i, always wins even when starting at a very low density, thus indicating that only the most selfish strategy is an ESS. For two altruistic strategies, coexistence is again possible but the coexistence region of the spatial game is significantly smaller than that of the replicator equation: except in the one-dimensional nearest neighbor case, coexistence occurs in a thorn-shaped region starting at the bifurcation point $a_1 = a_2 = 0$. The smaller the range of the interactions and the spatial dimension, the smaller the coexistence region. In the one-dimensional nearest neighbor case, the simulations are particularly difficult to interpret when

$$a_{11} + a_{12} \ < \ 0 \ < \ a_{12} \quad \text{and} \quad a_{22} + a_{21} \ < \ 0 \ < \ a_{21}. \tag{4}$$

See Figure 3 for a picture of two realizations when (4) holds. However, we were able to prove that the one-dimensional nearest neighbor system clusters except in a parameter region with measure

zero in the space of the 2×2 matrices in which all the players have a zero payoff eventually, thus leading to a fixation of the system in a configuration in which both strategies are present. More generally, we conjecture that, except in this parameter region with measure zero, the least altruistic strategy always wins, just as in the presence of selfish-selfish interactions. The thick continuous lines on the right-hand side of Figures 1 and 2 summarize our conjectures for the spatial game in the one-dimensional nearest neighbor case and all the other cases, respectively. These results are reminiscent of the ones obtained for the models introduced in [10, 11, 15] which, though they are not examples of evolutionary games, also include density dependent birth or death rates. Our proofs and the proofs in these three references strongly differ while showing the same pattern: for all four models, the inclusion of local interactions induces a reduction of the coexistence region of the mean-field model and there is a dominant type that wins even when starting at arbitrarily low density in the region where the mean-field model displays bistability.

We now state our analytical results for the spatial stochastic process, which confirms in particular these two important aspects. To motivate and explain our first result, we observe that the presence of density dependent birth and death rates typically precludes the existence of a mathematically tractable dual process. Note however that if $a_{11} = a_{12}$ and $a_{22} = a_{21}$ then the payoff of players of either type is constant across all possible spatial configurations: birth and death rates are no longer density dependent. For this specific choice of the payoffs, the process reduces to a biased voter model, therefore strategy 1 wins if in addition $a_{12} > a_{21}$. For all other payoff matrices, the dynamics is more complicated but there is a large parameter region in which the spatial game can still be coupled with a biased voter model to deduce that strategy 1 wins. This parameter region is specified in the following theorem. See also the right-hand side of Figure 2 where the boundary of this region is represented in dashed lines on the $a_{11} - a_{22}$ plane.

Theorem 1 – *Assume that $a_{12} > a_{21}$. Then, strategy 1 wins whenever*

$$\max(a_{22}, a_{21}) \ + \ a_{21}\,(N-1)^{-1} \ < \ \min(a_{11}, a_{12}) \ + \ a_{12}\,(N-1)^{-1}.$$

This parameter region intersects the regions in which the replicator equation displays coexistence and bistability. In particular, the theorem confirms that the inclusion of local interactions induces a reduction of the coexistence and bistable regions in accordance with the simulation results. The next two results strengthen the previous theorem by proving that the parameter region in which there is a unique ESS extends to arbitrarily small/large values of the payoffs a_{11} and a_{22}. To state these results, it is convenient to introduce the vector $\bar a_{11} := (a_{12}, a_{21}, a_{22})$.

Theorem 2 – *For all $\bar a_{11}$ there exists $m(\bar a_{11}) < \infty$ such that*

$$\text{strategy 1 wins whenever } a_{11} > m(\bar a_{11}).$$

This implies that, in accordance with our numerical simulations, the parameter region in which the replicator equation is bistable while there is a unique ESS for the spatial game is much larger than the parameter region covered by Theorem 1. Note also that, in view of the symmetry of the model, the previous theorem also holds by exchanging the roles of the two strategies.

Theorem 3 – *Let $a_{12} < 0$. For all $\bar a_{11}$ there exists $m(\bar a_{11}) < \infty$ such that*

$$\text{strategy 2 wins whenever } a_{11} < -m(\bar a_{11}).$$

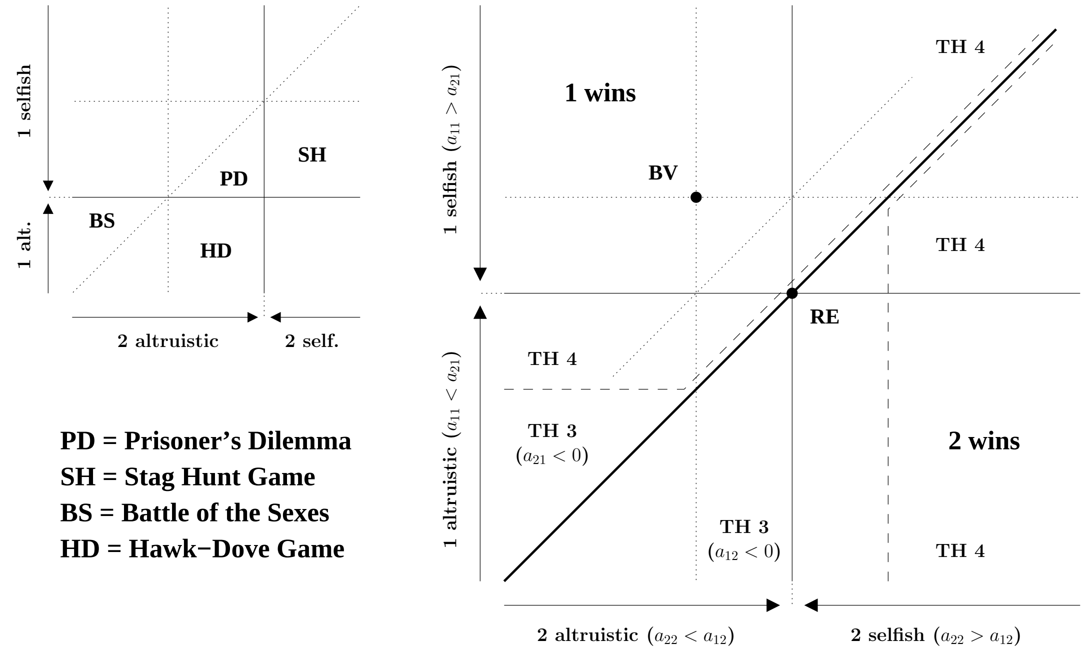


FIGURE 1. List of the most popular 2×2 games on the left and phase diagrams of the spatial game along with a summary of the theorems in the $a_{11} - a_{22}$ plane on the right. The thick lines refer to the transition curves suggested by the simulations. BV = biased voter model and RE = replicator equation.

This implies that, again in accordance with our numerical simulations, the parameter region in which coexistence occurs for the replicator equation while there is a unique ESS for the spatial game is much larger than the one covered by Theorem 1. Once more, we point out that, in view of the symmetry of the model, the theorem also holds by exchanging the roles of the two strategies as indicated in Figure 2. The previous two results hold regardless of the spatial dimension and the range of the interactions and can be significantly improved in the one-dimensional nearest neighbor case through an analysis of the boundaries of the system. More precisely, letting

$$\begin{aligned} \mathcal{M}_2^* \ := \ \{ & A = (a_{ij}) \text{ such that} \\ & a_{11}\, a_{12}\, a_{21}\, a_{22}\, (a_{11} + a_{12})(a_{22} + a_{21}) \neq 0 \text{ and } a_{11} + a_{12} \neq a_{22} + a_{21} \} \end{aligned} \tag{5}$$

we have the following theorem.

Theorem 4 – *Assume that $M = d = 1$. Then,*

- *strategy 1 wins for all $a_{11} > \max(a_{22}, a_{21}) + (a_{21} - a_{12})$ and*
- *the system clusters for all $A \in \mathcal{M}_2^*$.*

Figure 1 gives the phase diagram of the one-dimensional nearest neighbor process obtained by combining Theorems 3–4. The parameter region in the first part of Theorem 4 and the one obtained by symmetry are represented in dashed lines when $a_{12} > a_{21}$. For most of the parameter region in which at least one strategy is selfish, the most selfish strategy wins, which significantly improves Theorem 2. The second part of the theorem supplements Theorem 3 by proving that, except in the

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$M=1$	1.0000	1.4000	1.4706	1.4906	1.4969	1.4990	1.4997	1.4999	1.5000
$M=2$	1.3333	1.6140	1.6566	1.6647	1.6663	1.6666	1.6667	1.6667	1.6667
$M=3$	1.5000	1.7195	1.7457	1.7494	1.7499	1.7500	1.7500	1.7500	1.7500
$M=4$	1.6000	1.7803	1.7978	1.7998	1.8000	1.8000	1.8000	1.8000	1.8000
$M=5$	1.6667	1.8196	1.8321	1.8332	1.8333	1.8333	1.8333	1.8333	1.8333
$M=6$	1.7143	1.8470	1.8564	1.8571	1.8571	1.8571	1.8571	1.8571	1.8571
$M=7$	1.7500	1.8672	1.8745	1.8750	1.8750	1.8750	1.8750	1.8750	1.8750
$M=8$	1.7778	1.8827	1.8885	1.8889	1.8889	1.8889	1.8889	1.8889	1.8889
$M=9$	1.8000	1.8950	1.8997	1.9000	1.9000	1.9000	1.9000	1.9000	1.9000

TABLE 1
$c(M,d)$ for different values of the range M and the dimension d.

measure zero parameter region that corresponds to $A \in \mathcal{M}_2^*$, coexistence is not possible in the one-dimensional nearest neighbor case. Finally, our last theorem looks more closely at the interactions between two altruistic strategies and confirms that, except in the one-dimensional nearest neighbor case, coexistence is possible for the spatial game.

Theorem 5 – *There is $m := m(a_{12}, a_{21}) > 0$ such that coexistence occurs when*

$$c(M,d)\, a_{22} < a_{11} < -m \quad \text{and} \quad c(M,d)\, a_{11} < a_{22} < -m$$

where, for each range M and spatial dimension d,

$$c(M,d) \ := \ \frac{2M\,((2M+1)^d - 2)}{(M+1)\,(2M\,(2M+1)^{d-1} - 1)}. \tag{6}$$

Table 1 and the theorem indicate that, except in the one-dimensional nearest neighbor case in which the region given by the theorem is empty because $c(1,1) = 1$, the coexistence region contains an infinite subset of a certain triangle whose range increases with respect to (6). The larger the range of the interactions and the spatial dimension, the larger this triangle. We refer to Figure 2 for a summary of the theorems that exclude the one-dimensional nearest neighbor case.

The role of space in the most popular games. The last step before going into the details of the proofs is to discuss the implications of our results in the most popular symmetric two-person games. To define these games, note that there are $4! = 24$ possible orderings of the four payoffs therefore, also accounting for symmetry, twelve possible strategic situations corresponding to twelve symmetric two-person games involving two strategies. These twelve regions of the parameter space are represented in the $a_{11} - a_{22}$ plane on the left-hand diagrams of Figures 1–2 along with the names of the most popular games under the assumption $a_{12} > a_{21}$.

Prisoner's Dilemma – The prisoner's dilemma is probably the most popular symmetric two-person game. When $a_{12} > a_{21}$, strategy 1 means defection whereas strategy 2 means cooperation. From the point of view of the replicator equation, defection is the only ESS. Numerical simulations suggest that the same holds for our spatial model, which is covered in part in the general case and completely in the one-dimensional nearest neighbor case in Theorems 1 and 4.

Stag Hunt – In the stag hunt game with $a_{12} > a_{21}$, strategy 1 represents safety: hunting a hare, whereas strategy 2 represents social cooperation: hunting a stag. In the absence of space, both strategies are evolutionary stable. In contrast, Theorem 2 shows that, in the presence of local

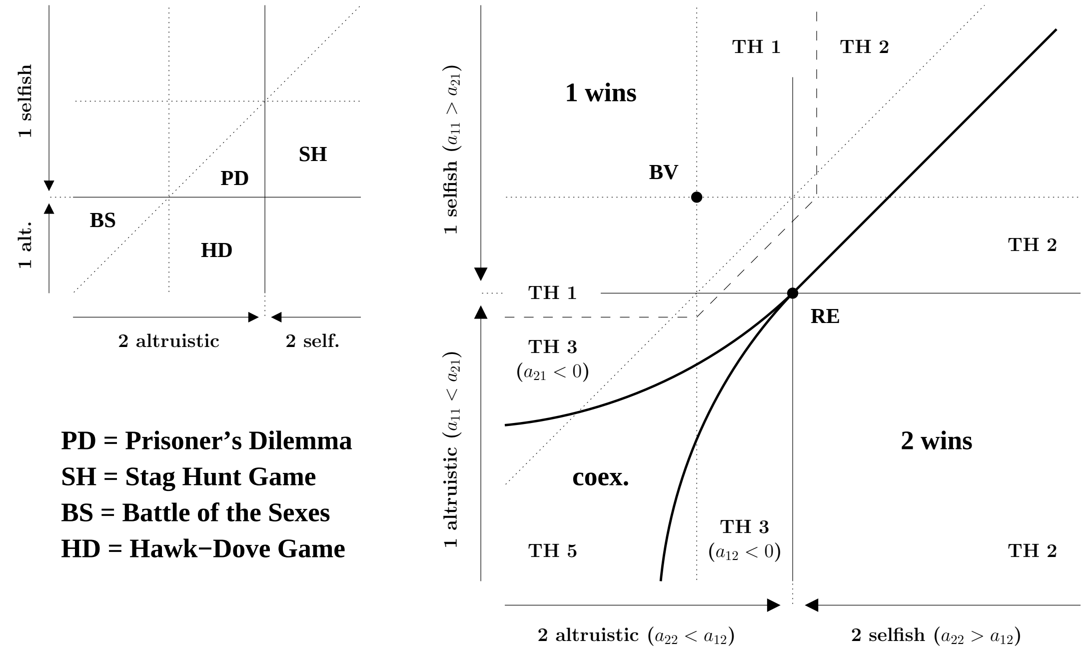


FIGURE 2. List of the most popular 2×2 games on the left and phase diagrams of the spatial game along with a summary of the theorems in the $a_{11} - a_{22}$ plane on the right. The thick lines refer to the transition curves suggested by the simulations. BV = biased voter model and RE = replicator equation.

interactions, social cooperation is the only ESS if the reward a_{22} for social cooperation is high enough, i.e., a stag is worth much more than a hare, whereas if the reward is not significant then safety becomes the only ESS according to Theorems 1 and 4.

Hawk-Dove – In the hawk-dove game, strategy 1 represents hawks that fight for the resource and strategy 2 doves that share peacefully the resource. The cost of a fight is larger than the value of the contested resource, which makes this game an example of anti-coordination game: the best possible response to a strategy is to play the other strategy. In the absence of space, none of the strategies is evolutionary stable so coexistence occurs. In contrast, Theorem 3 indicates that, in the presence of a spatial structure, the dove strategy is the only ESS when the cost of an escalated fight is high enough or equivalently when a_{11} is small enough.

Battle of the Sexes – In the battle of the sexes, husband and wife cannot remember if they planned to meet at the opera or at the football match. The husband would prefer the match and the wife the opera, but overall both would prefer to go to the same place. Mutual cooperation, i.e., both go to the place that the other prefers, leads to the lowest possible payoff a_{22} which makes this game another example of anti-coordination game. In the absence of space, none of the strategies is evolutionary stable so coexistence occurs. Coexistence is also possible in the presence of a spatial structure according to Theorem 5. However, if the cost of mutual cooperation is too high, i.e., a_{22} too small, then defection becomes the unique ESS according to Theorem 3.

The rest of the paper is devoted to the proofs of the theorems. We point out that the theorems are not proved in the order they are stated but instead grouped based on the approach and techniques they rely on, which makes the reading of the proofs somewhat easier.

2. Proof of Theorem 4

We first study the one-dimensional nearest neighbor system. The analysis in this case relies on the study of the process that keeps track of the boundaries between the two strategies and strongly differs from the analysis of the system in higher spatial dimensions or with a larger range of interactions. Throughout this section, we let ξ_t denote the **boundary process**

$$\xi_t(x) \ := \ \eta_t(x+1/2) - \eta_t(x-1/2) \quad \text{for all} \quad x \in \mathbb{D} := \mathbb{Z} + 1/2$$

and think of sites in state 0 as empty and sites in state ± 1 as occupied by a $\pm$ particle. To begin with, we assume that $\eta_0(x) = 1$ for all $x \leq 0$, i.e., from the point of view of the boundary process, we start with no particle to the left of the origin, and let

$$X_t \ := \ \inf \{x \in \mathbb{D} : \xi_t(x) \neq 0\} \ = \ \inf \{x \in \mathbb{D} : \xi_t(x) = 1\}$$

denote the position of the leftmost particle, which is necessarily a + particle in view of the initial configuration. The key to proving the first part of Theorem 4 is given by the next lemma, which shows the result when starting from this particular configuration.

Lemma 6 – *Assume that* $a_{11} > \max(a_{22}, a_{21}) + (a_{21} - a_{12})$. *Then,* $\lim_{t \to \infty} X_t = \infty$.

PROOF. It suffices to prove that the expected value of the drift

$$E(D_t) \ := \ \lim_{h \to 0} h^{-1} E(X_{t+h} - X_t) \ > \ 0.$$

Since the expected value of the drift depends on the distance from site X_t of the next particle, which is necessarily a $-$ particle, we also introduce the gap process

$$\begin{aligned} K_t \ &:= \ \inf \{x \in \mathbb{D} : x > X_t \text{ and } \xi_t(x) \neq 0\} - X_t \\ &= \ \inf \{x \in \mathbb{D} : x > X_t \text{ and } \xi_t(x) = -1\} - X_t. \end{aligned}$$

Looking at the payoff of the players at sites $X_t \pm 1/2$, we find

$$\begin{aligned} E(D_t \,|\, K_t \geq 2) \ &= \ \max(0, a_{11} + a_{12}) + \max(0, -a_{22} - a_{21}) - \max(0, -a_{11} - a_{12}) \\ &\qquad - \max(0, a_{22} + a_{21}) \ = \ (a_{11} + a_{12}) - (a_{22} + a_{21}) \\ E(D_t \,|\, K_t = 1) \ &\geq \ 2 \times \max(0, a_{11} + a_{12}) + 4 \times \max(0, -2a_{21}) \\ &\qquad - \max(0, -a_{11} - a_{12}) - \max(0, 2a_{21}) \\ &\geq \ \max(0, a_{11} + a_{12}) - \max(0, -a_{11} - a_{12}) + \max(0, -2a_{21}) \\ &\qquad - \max(0, 2a_{21}) \ = \ a_{11} + a_{12} - 2a_{21}. \end{aligned}$$

In particular, for all $a_{11} > \max(a_{22}, a_{21}) + (a_{21} - a_{12})$, we have

$$\begin{aligned} E(D_t) \ &\geq \ \min(a_{11} + a_{12} - a_{22} - a_{21}, \, a_{11} + a_{12} - 2a_{21}) \\ &= \ a_{11} - (a_{21} - a_{12}) - \max(a_{22}, a_{21}) \ > \ 0. \end{aligned}$$

This completes the proof. $\square$

To deduce the first part of the theorem, note that the proof of the lemma implies more generally that, with positive probability, the leftmost + particle never crosses site zero:

$$P\,(X_t > 0 \text{ for all } t > 0) \ > \ 0.$$

In particular, starting with two neighbors following strategy 1, we have

$$\begin{aligned} & P\,(\lim_{t \to \infty} \eta_t(x) = 1 \text{ for all } x \in \mathbb{Z}) \\ & \quad \geq \ P\,(X_t > 0 \text{ for all } t > 0)\ P\,(-X_t < 0 \text{ for all } t > 0) \ > \ 0. \end{aligned}$$

We easily deduce that starting from a product measure with a positive density of type 1 players, strategy 1 wins with probability one, which proves the first part of the theorem. The next two lemmas focus on the second part of the theorem whose proof consists in showing extinction of the boundary process starting from any initial configuration.

Lemma 7 – *Assume that $A \in \mathcal{M}_2^*$ as in* (5)*. Then, the system clusters if*

$$a_{22} + a_{21} < a_{11} + a_{12} \quad \textit{and} \quad (a_{11} + a_{12} > 0 \quad \textit{or} \quad a_{21} < 0). \tag{7}$$

PROOF. As pointed out before the statement of the lemma, in view of the definition of the boundary process and the definition of clustering, it suffices to prove that

$$u(t) \ := \ P\,(\xi_t(x) \neq 0) \ \to \ 0 \quad \text{as} \quad t \to \infty.$$

Due to translation invariance of the initial distribution and the evolution rules, the probability above is indeed constant across space. The definition of the boundary process also implies that two consecutive particles must have opposite signs. Moreover, due to one-dimensional nearest neighbor interactions, particle cannot be created and if a particle jumps onto another particle then both particles, necessarily with opposite signs, annihilate. In particular,

$$u'(t) \leq 0 \quad \text{therefore} \quad \lim_{t \to \infty} u(t) := l \ \text{ exists.}$$

To prove that the limit $l = 0$, we must prove that, as long as the density of particles is positive, annihilating events occur so the density is strictly decreasing. Having a time s at which the density of particles is positive, we consider an arbitrary + particle alive at time s and the next $-$ particle on its right. Let $X_t^{\pm}$ denote the position of the $\pm$ particle at time t, which is well defined until the particle is killed when we set $X_t^{\pm} = \varnothing$. Also, we define

$$\tau_+ \ := \ \inf\{t > s : X_t^+ = \varnothing\} \quad \text{and} \quad \tau_- \ := \ \inf\{t > s : X_t^- = \varnothing\}.$$

We claim that $\inf\,(\tau_+, \tau_-) < \infty$, which indeed implies that the density of particles decreases as long as it is positive and that it converges to zero. To prove our claim, let

$$\begin{aligned} \sigma_+ \ &:= \ \text{time at which the + particle at } X_t^+ \text{ annihilates with a } - \text{ particle on its left} \\ \sigma_- \ &:= \ \text{time at which the } - \text{ particle at } X_t^- \text{ annihilates with a + particle on its right.} \end{aligned}$$

By inclusion of events, we have

$$P\,(\inf\,(\tau_+, \tau_-) < \infty \mid \inf\,(\sigma_+, \sigma_-) < \infty) \ = \ 1. \tag{8}$$

Moreover, in view of the first inequality in (7), each + particle with no particle in its neighborhood has a positive drift: more precisely, for the + particle at X_t we have

$$\begin{aligned}
&\lim_{h \to 0} h^{-1} P(X_{t+h}^+ - X_t^+ = 1 \,|\, X_t^+ \neq \varnothing \text{ and } \xi_t(X_t^+ - 1) = \xi_t(X_t^+ + 1) = 0) \\
&\quad = (1/2)(\max(0, a_{11} + a_{12}) + \max(0, -a_{22} - a_{21})) \\
&\quad > (1/2)(\max(0, a_{11} + a_{12}) - (a_{11} + a_{12}) + \max(0, -a_{22} - a_{21}) + (a_{22} + a_{21})) \\
&\quad = (1/2)(\max(0, -a_{11} - a_{12}) + \max(0, a_{22} + a_{21})) \\
&\quad = \lim_{h \to 0} h^{-1} P(X_{t+h}^+ - X_t^+ = -1 \,|\, X_t^+ \neq \varnothing \text{ and } \xi_t(X_t^+ - 1) = \xi_t(X_t^+ + 1) = 0).
\end{aligned}$$

Since on the event $\sigma_+ = \infty$ the particle at X_t^+ cannot jump onto a $-$ particle on its left, we deduce that, on this event, the position of the particle has a positive drift until it is one unit from the $-$ particle on its right. Similarly, on the event $\sigma_- = \infty$, the position of the particle at X_t^- has a negative drift until it is one unit from the + particle on its left. This implies that

$$\begin{aligned}
&P(X_t^- - X_t^+ = 1 \text{ and} \\
&\qquad \xi_t(X_t^- + 1)\, \xi_t(X_t^+ - 1) = 0 \text{ for some } t > s \,|\, \inf(\sigma_+, \sigma_-) = \infty) \;=\; 1.
\end{aligned} \tag{9}$$

Also, each time $X_t^- - X_t^+ = 1$, both particles annihilate at rate at least

$$\max(0, -a_{21}) + (1/2) \max(0, a_{11} + a_{12}).$$

The second inequality in (7) implies that this rate is strictly positive which, together with (9) and a basic restart argument, further implies that

$$P(\inf(\tau_+, \tau_-) < \infty \,|\, \inf(\sigma_+, \sigma_-) = \infty) \;=\; 1. \tag{10}$$

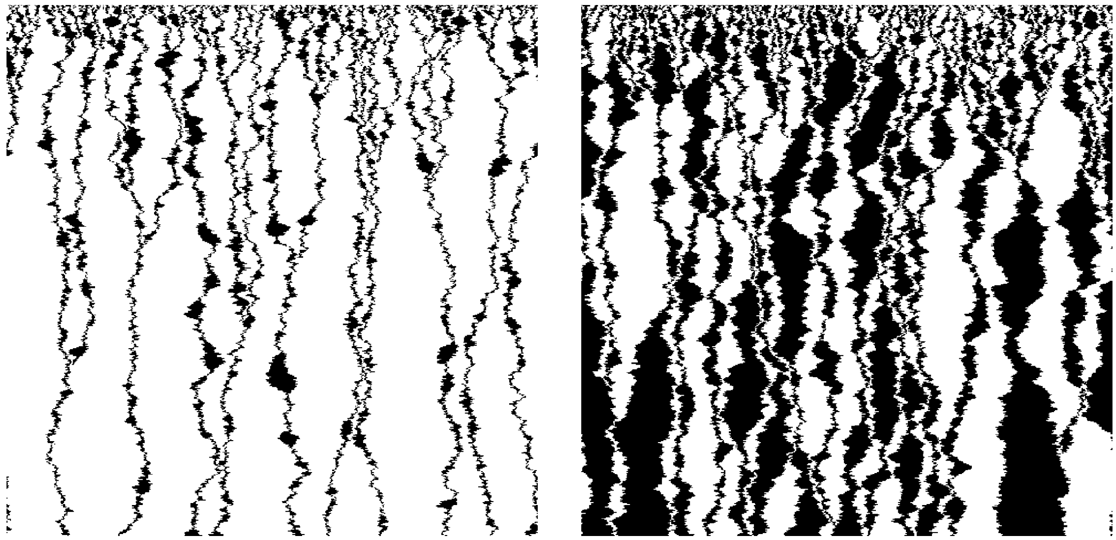
The lemma directly follows from the combination of (8) and (10). $\square$

To complete the proof of the theorem, the last step is to prove the analog of Lemma 7 when the second set of inequalities in (7) does not hold. This includes in particular all the payoff matrices that satisfy (4). This case is rather delicate since a player of either type cannot change her strategy whenever her two nearest neighbors and next two nearest neighbors all four follow the same strategy. In particular, two particles next to each other annihilate at a positive rate only if there is a third particle nearby so the idea of the proof is to show that we can indeed bring sets of three consecutive particles together. Figure 3 gives an illustration of this problem: boundaries by pair repulse each other and at least three particles are necessary to induce annihilation.

Lemma 8 – *Assume that $A \in \mathcal{M}_2^*$. Then, the system clusters if*

$$a_{22} + a_{21} < a_{11} + a_{12} < 0 \quad \text{and} \quad a_{21} > 0. \tag{11}$$

Proof. Following the same approach as in the previous lemma, it suffices to prove that, starting with a positive density of boundaries, annihilating events occur in a finite time within a given finite set of consecutive boundaries. The main difficulty is that condition (11) now implies that starting with a single type 2 player, the two resulting boundaries cannot annihilate therefore to prove the occurrence of annihilating events, we need to look at a set of four boundaries instead of two like in the proof of the previous lemma. To begin with, we start from a configuration with infinitely many



(a) $A = ((-8,3),(4,-8))$ (b) $A = ((-8,4),(4,-8))$

FIGURE 3. Realizations of the one-dimensional nearest neighbor spatial game on the torus $\mathbb{Z}/600\mathbb{Z}$ for two different payoff matrices that satisfy the inequalities in (4). Time goes down until time 10,000.

type 1 players and exactly four boundaries, which forces the initial number of type 2 players to be finite, and denote the position of the boundaries by

$$X_t^+ \ < \ X_t^- \ < \ Y_t^+ \ < \ Y_t^-$$

before an annihilating event has occurred. The same argument as in the proof of the previous lemma based on the first inequality in (11) implies that

$$\begin{array}{r} \lim_{h \to 0} \ h^{-1} \, E \, ((X_{t+h}^- - X_{t+h}^+) - (X_t^- - X_t^+) \, | \\ X_t^- - X_t^+ > 1 \text{ and } Y_t^+ - X_t^- > 1) \ < \ 0. \end{array} \tag{12}$$

The same applies to the two rightmost boundaries:

$$\begin{array}{r} \lim_{h \to 0} \ h^{-1} \, E \, ((Y_{t+h}^- - Y_{t+h}^+) - (Y_t^- - Y_t^+) \, | \\ Y_t^- - Y_t^+ > 1 \text{ and } Y_t^+ - X_t^- > 1) \ < \ 0. \end{array} \tag{13}$$

Moreover, by symmetry, we have

$$\begin{array}{l} \lim_{h \to 0} \ h^{-1} \, P \, ((X_{t+h}^- + X_{t+h}^+) - (X_t^- + X_t^+) = 1 \, | \, Y_t^+ - X_t^- > 1) \\ \quad = \ \lim_{h \to 0} \ h^{-1} \, P \, ((X_{t+h}^- + X_{t+h}^+) - (X_t^- + X_t^+) = -1 \, | \, Y_t^+ - X_t^- > 1). \end{array} \tag{14}$$

In words, the midpoint between the two leftmost boundaries evolve according to a symmetric random walk. The same holds for the midpoint between the two rightmost boundaries. To deduce the occurrence of an annihilating event, we distinguish two cases:

- **Case 1** – Assume (11) and $a_{12} < 0$. In this case, (14) and the recurrence of one-dimensional symmetric simple random walks imply that

$$P \, (Y_t^+ - X_t^- = 1 \text{ for some } t > 0) \ = \ 1.$$

Since the event above induces a configuration in which a type 1 player has two type 2 neighbors, and so a negative payoff $a_{12} < 0$, each time this event occurs, the two intermediate boundaries annihilate at a positive rate. This, together with a basic restart argument, implies the occurrence of an annihilating event after an almost surely finite time.

- **Case 2** – Assume (11) and $a_{12} > 0$. In this case, (12)–(14) imply that, with probability one, we can bring three consecutive boundaries together: more precisely,

$$P\,(Y_t^+ - X_t^- = 1 \text{ and } (X_t^- - X_t^+ = 1 \text{ or } Y_t^- - Y_t^+ = 1) \text{ for some } t > 0) \ = \ 1.$$

Since the event above induces a configuration in which a type 2 player has two type 1 neighbors, and so a positive payoff $a_{21} > 0$, each time this event occurs, either the two leftmost boundaries or the two rightmost boundaries annihilate at a positive rate. We again deduce the occurrence of an annihilating event after an almost surely finite time.

The two results above still hold when starting from a configuration with a positive density of boundaries unless the leftmost of the four boundaries or the rightmost of the four boundaries annihilate before with another boundary. In any case, each set of four consecutive boundaries is reduced by one after an almost surely finite time. The lemma follows. □

Lemmas 7–8 imply clustering for all $A \in \mathcal{M}_2^*$ with $a_{11} + a_{12} > a_{22} + a_{21}$. The second part of the theorem directly follows by also using some obvious symmetry.

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1, which relies on a standard coupling argument between the spatial game and a biased voter model that favors individuals of type 1. Recall that the biased voter model is the spin system with flip rate

$$c_{BV}(x,\xi) \ = \ \mu_1\, f_1(x,\xi)\,\mathbf{1}\{\xi(x) = 2\} + \mu_2\, f_2(x,\xi)\,\mathbf{1}\{\xi(x) = 1\}$$

for which individuals of type 1 win whenever $\mu_1 > \mu_2$. Recall also that the spatial game reduces to such a spin system if and only if the payoff received by players of either type is constant regardless of the spatial configuration. In particular, strategy 1 wins whenever

$$a_{11} \ = \ a_{12} \ > \ a_{21} \ = \ a_{22}.$$

For all other parameters, the dynamics is more complicated but the process can be coupled with a biased voter model that favors type 1 individuals in a certain parameter region. To make this argument rigorous and prove Theorem 1, we introduce the payoff functions

$$\begin{array}{rcl} \phi_1(z) & := & a_{12}\,(z/N) + a_{11}\,(1 - z/N) \ = \ (a_{12} - a_{11})(z/N) + a_{11} \\ \phi_2(z) & := & a_{22}\,(z/N) + a_{21}\,(1 - z/N) \ = \ (a_{22} - a_{21})(z/N) + a_{21}. \end{array}$$

The coupling argument is given in the proof of the following lemma.

Lemma 9 – *Assume that $a_{12} > a_{21}$. Then, strategy 1 wins whenever*

$$\max\,(\phi_2(z) : z \in \{0, 1, \ldots, N-1\}) \ < \ \min\,(\phi_1(z) : z \in \{1, 2, \ldots, N\}). \tag{15}$$

PROOF. Denoting by $c_{SG}(x, \eta)$ the flip rate of the spatial game, we have

$$c_{SG}(x, \eta) \ = \ c_{BV}(x, \xi) \ = \ 0 \quad \text{when} \quad f_{\eta(x)}(x, \eta) \ = \ f_{\xi(x)}(x, \xi) \ = \ 1. \tag{16}$$

Now, observe that the player at x may flip $2 \to 1$ because she has a negative payoff and so a positive death rate or because she has a neighbor following strategy 1 that has a positive payoff and so a positive birth rate. In particular, given that the player at vertex x follows strategy 2 and has at least one neighbor following strategy 1, the rate at which the strategy at x flips is

$$\begin{array}{rcl} c_{SG}(x, \eta) & := & c_{SG}(x, \eta \,|\, \eta(x) = 2 \text{ and } f_1(x, \eta) \neq 0) \\ & = & \max(0, -\phi(x, \eta))\, f_1(x, \eta) \\ & & \quad + \ N^{-1} \ \sum_{y \sim x} \max(0, \phi(y, \eta))\, \mathbf{1}\{\eta(y) = 1\} \\ & \geq & \min_{z \neq N} \ \max(0, -\phi_2(z))\, f_1(x, \eta) \\ & & \quad + \ \min_{z \neq 0} \ \max(0, \phi_1(z))\, f_1(x, \eta). \end{array} \tag{17}$$

Similarly, given that the player at vertex x follows strategy 1 and has at least one neighbor following strategy 2, the rate at which the strategy at x flips is

$$\begin{array}{rcl} c_{SG}(x, \eta) & := & c_{SG}(x, \eta \,|\, \eta(x) = 1 \text{ and } f_2(x, \eta) \neq 0) \\ & = & \max(0, -\phi(x, \eta))\, f_2(x, \eta) \\ & & \quad + \ N^{-1} \ \sum_{y \sim x} \max(0, \phi(y, \eta))\, \mathbf{1}\{\eta(y) = 2\} \\ & \leq & \max_{z \neq 0} \ \max(0, -\phi_1(z))\, f_2(x, \eta) \\ & & \quad + \ \max_{z \neq N} \ \max(0, \phi_2(z))\, f_2(x, \eta). \end{array} \tag{18}$$

Combining (16)-(18), we obtain that strategy 1 wins whenever

$$\begin{array}{rl} \mu_2 \ := & \max_{z \neq 0} \ \max(0, -\phi_1(z)) + \max_{z \neq N} \ \max(0, \phi_2(z)) \\ & \quad < \ \min_{z \neq N} \ \max(0, -\phi_2(z)) + \min_{z \neq 0} \ \max(0, \phi_1(z)) \ =: \ \mu_1 \end{array} \tag{19}$$

since, under this assumption, if $\eta(x) \leq \xi(x)$ for all $x \in \mathbb{Z}^d$ then

$$\begin{array}{rcl} c_{SG}(x, \eta) \ \leq \ \mu_2\, f_2(x, \xi) & = & c_{BV}(x, \xi) \quad \text{when} \quad \eta(x) = \xi(x) = 1 \\ c_{SG}(x, \eta) \ \geq \ \mu_1\, f_1(x, \xi) & = & c_{BV}(x, \xi) \quad \text{when} \quad \eta(x) = \xi(x) = 2 \end{array}$$

which, according to Theorem III.1.5 in [12], implies that the set of type 1 players dominates stochastically its counterpart in a biased voter model that favors type 1. To complete the proof, it remains to show that (15) implies (19). Note that (19) is equivalent to

$$\begin{array}{l} \max_{z \neq N} \ \max(0, \phi_2(z)) \ - \ \min_{z \neq N} \ \max(0, -\phi_2(z)) \\ \qquad < \ \min_{z \neq 0} \ \max(0, \phi_1(z)) \ - \ \max_{z \neq 0} \ \max(0, -\phi_1(z)). \end{array} \tag{20}$$

Note also that the left-hand side of (20) reduces to

$$\begin{array}{l} \max_{z \neq N} \ \max(0, \phi_2(z)) \ - \ \min_{z \neq N} \ \max(0, -\phi_2(z)) \\ \quad = \ \max(0, \max_{z \neq N} \ \phi_2(z)) \ - \ \max(0, \min_{z \neq N} (-\phi_2(z))) \\ \quad = \ \max(0, \max_{z \neq N} \ \phi_2(z)) \ - \ \max(0, -\max_{z \neq N} \ \phi_2(z)) \\ \quad = \ \max(0, \max_{z \neq N} \ \phi_2(z)) \ + \ \min(0, \max_{z \neq N} \ \phi_2(z)) \ = \ \max_{z \neq N} \ \phi_2(z). \end{array} \tag{21}$$

Similarly, for the payoff of type 1 players, we have

$$\min_{z \neq 0} \ \max(0, \phi_1(z)) \ - \ \max_{z \neq 0} \ \max(0, -\phi_1(z)) \ = \ \min_{z \neq 0} \ \phi_1(z). \tag{22}$$

Since (15), (21) and (22) imply (20) and then (19), the proof is complete. □

In the following lemma, we complete the proof of Theorem 1 based on (15).

Lemma 10 – *Assume that $a_{12} > a_{21}$. Then, strategy 1 wins whenever*

$$\max(a_{22}, a_{21}) \ + \ a_{21} (N-1)^{-1} \ < \ \min(a_{11}, a_{12}) \ + \ a_{12} (N-1)^{-1}. \tag{23}$$

Proof. This directly follows from Lemma 9 by showing that the parameter region in which the inequality in (15) holds is exactly (23). To re-write (15) explicitly in terms of the payoffs, we distinguish four cases depending on the monotonicity of the functions ϕ_1 and ϕ_2.

- **Case 1** – When $\max(a_{22}, a_{21}) = a_{21}$ and $\min(a_{11}, a_{12}) = a_{12}$, both payoff functions are decreasing therefore, according to the previous lemma, strategy 1 wins whenever

$$\max_{z \neq N} \ \phi_2(z) \ = \ \phi_2(0) \ = \ a_{21} \ < \ a_{12} \ = \ \phi_1(N) \ = \ \min_{z \neq 0} \ \phi_1(z)$$

  which is always true under our general assumption $a_{12} > a_{21}$.

- **Case 2** – When $\max(a_{22}, a_{21}) = a_{21}$ and $\min(a_{11}, a_{12}) = a_{11}$, according to the previous lemma, strategy 1 wins whenever

$$\begin{aligned} \max_{z \neq N} \ \phi_2(z) \ &= \ \phi_2(0) \ = \ a_{21} \ = \ (1 - 1/N) \max(a_{22}, a_{21}) + (1/N)\, a_{21} \\ &< \ (1 - 1/N) \min(a_{11}, a_{12}) + (1/N)\, a_{12} \\ &= \ (1 - 1/N)\, a_{11} + (1/N)\, a_{12} \ = \ \phi_1(1) \ = \ \min_{z \neq 0} \ \phi_1(z) \end{aligned}$$

  which is true whenever (23) holds.

- **Case 3** – When $\max(a_{22}, a_{21}) = a_{22}$ and $\min(a_{11}, a_{12}) = a_{12}$, according to the previous lemma, strategy 1 wins whenever

$$\begin{aligned} \max_{z \neq N} \ \phi_2(z) \ &= \ \phi_2(N-1) \ = \ (1 - 1/N)\, a_{22} + (1/N)\, a_{21} \\ &= \ (1 - 1/N) \max(a_{22}, a_{21}) + (1/N)\, a_{21} \\ &< \ (1 - 1/N) \min(a_{11}, a_{12}) + (1/N)\, a_{12} \ = \ a_{12} \ = \ \phi_1(N) \ = \ \min_{z \neq 0} \ \phi_1(z) \end{aligned}$$

  which is true whenever (23) holds.

- **Case 4** – When $\max(a_{22}, a_{21}) = a_{22}$ and $\min(a_{11}, a_{12}) = a_{11}$, according to the previous lemma, strategy 1 wins whenever

$$\begin{aligned} \max_{z \neq N} \ \phi_2(z) \ &= \ \phi_2(N-1) \ = \ (1 - 1/N)\, a_{22} + (1/N)\, a_{21} \\ &= \ (1 - 1/N) \max(a_{22}, a_{21}) + (1/N)\, a_{21} \ < \ (1 - 1/N) \min(a_{11}, a_{12}) + (1/N)\, a_{12} \\ &= \ (1 - 1/N)\, a_{11} + (1/N)\, a_{12} \ = \ \phi_1(1) \ = \ \min_{z \neq 0} \ \phi_1(z) \end{aligned}$$

  which is true whenever (23) holds.

This completes the proof of the lemma and the proof of Theorem 1. □

4. Proof of Theorem 5

The common background behind the proofs of the remaining three theorems is the use of a block construction, though the arguments required to indeed be able to apply this technique strongly differ among these theorems. The idea of the block construction is to couple a certain collection of good events related to the process properly rescaled in space and time with the set of open sites of oriented site percolation on the directed graph $\mathcal{H}_1$ with vertex set

$$H \ := \ \{(z,n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : z_1 + z_2 + \cdots z_d + n \text{ is even}\}$$

and in which there is an oriented edge

$$\begin{aligned}(z,n) \to (z',n') \quad & \text{if and only if} \\ & z' = z \pm e_i \text{ for some } i = 1,2,\ldots,d \ \text{ and } \ n' = n+1\end{aligned}$$

where e_i is the ith unit vector. See the left-hand side of Figure 6 for a picture in $d = 1$. For a definition of oriented site percolation, we refer to Durrett [6] where the block construction is also reviewed in detail and employed to study different spatial processes. The existence of couplings between the spatial game and oriented percolation relies, among other things, on the application of Theorem 4.3 in [6] which requires certain good events to be measurable with respect to a so-called graphical representation of the process. Therefore, we need to construct the spatial game from a graphical representation, though we will not use it explicitly except in the last section. To construct the process graphically, we first introduce the maximum update rate:

$$\mathfrak{m} \ := \ \max_{i,j} \ |a_{ij}|.$$

Then, for all $x \in \mathbb{Z}^d$ and $n > 0$,

- we let $T_n(x) =$ the nth arrival time of a Poisson process with rate $\mathfrak{m}$
- we let $U_n(x) =$ a uniform random variable over the interval $(0, \mathfrak{m})$
- we let $V_n(x) =$ a uniform random variable over the interaction neighborhood N_x.

At the arrival times $T_n(x)$, we draw

$$\text{an arrow} \quad V_n(x) \to x \quad \text{with the label} \quad U_n(x) \tag{24}$$

and say that this arrow is **active** whenever

$$U_n(x) < |\phi(x, \eta_{t-})| \quad \text{where} \quad t := T_n(x).$$

Given an initial configuration, an argument due to Harris [7] implies that the process can be constructed going forward in time by setting

$$\eta_t(x) \ = \ \eta_t(V_n(x)) \ := \ \begin{cases} \eta_{t-}(x) & \text{when} \quad \phi(x, \eta_{t-}) > 0 \ \text{ and (24) is active} \\ \eta_{t-}(V_n(x)) & \text{when} \quad \phi(x, \eta_{t-}) < 0 \ \text{ and (24) is active} \end{cases}$$

where again $t := T_n(x)$. In case the arrow in (24) is not active, the update is canceled. In order to simplify a little bit some cumbersome expressions in the proofs of the remaining three theorems,

we identify from now on the spatial game with the set of the type 1 players, which is a common approach to study spin systems. We now focus on the proof of our coexistence result. The first step is to establish survival of the type 1 players when

$$(M,d) \neq (1,1) \quad \text{and} \quad a_{12} = a_{21} = 0 \quad \text{and} \quad c(M,d)\, a_{22} < a_{11} < -1 \tag{25}$$

which is done by comparing the spatial game ξ_t with $a_{12} = a_{21} = 0$ and one dependent oriented site percolation. We declare site $(z,n) \in H$ to be **occupied** whenever

$$\xi_{cnK} \cap B_2(Kz, 3K/5) \;\neq\; \varnothing \tag{26}$$

where $c > 0$ is a constant and K a large integer that will be fixed later and where $B_2(x,r)$ is the Euclidean ball with center x and radius r. Also we set

$$\mathbb{X}_n \;:=\; \{z \in \mathbb{Z}^d : (z,n) \in H \text{ and is occupied}\}. \tag{27}$$

In view of Theorem 4.3 in Durrett [6] and the fact that the spatial game is translation invariant in space and time, to prove that the process $\mathbb{X}_n$ dominates stochastically supercritical oriented site percolation, the main step is to show that the conditional probability

$$P\,((e_1,1) \text{ is occupied} \mid (0,0) \text{ is occupied})$$

can be made arbitrarily close to one by choosing K sufficiently large. To estimate this conditional probability, we start with a single type 1 player at site 0 and keep track of a tagged player of type 1 that moves to the target Ke_1. Denote by

$$r_t \;:=\; \max\{\pi_1(x) : x \in \xi_t\} \quad \text{and} \quad R_t \;:=\; \{x \in \xi_t : \pi_1(x) = r_t\}$$

the first coordinate of the rightmost type 1 players and the set of the rightmost type 1 players, respectively. Since $a_{12} = a_{21} = 0$, this set is always nonempty. In one dimension, it reduces to a singleton whereas in higher dimensions it may have more sites. In any case, we let X_t be the position of one of the rightmost type 1 players chosen uniformly at random among the ones who are the closest to the first axis, and call this player the **tagged player**. The key to proving that the set of type 1 players spreads in the direction of e_1 is given by the next lemma.

Lemma 11 – *Assume* (25)*. Then,*

$$\mathcal{D}_t \;:=\; \lim_{h \to 0} h^{-1}\, E\,(X_{t+h} - X_t \,|\, \xi_t) \;>\; 0 \quad \textit{almost surely.}$$

PROOF. To begin with, we introduce the process

$$L_t \;:=\; \inf\{\pi_1(X_t - x) : x \in \xi_t \text{ and } x \neq X_t\}.$$

In words, L_t is the distance along the first axis between the tagged player and the second rightmost player of type 1, which is also the length of a jump to the left at the time the tagged player changes her strategy. See Figure 4 for a picture. First, we assume that $L_t = L \in [1,M]$. On this event, the fraction of type 1 neighbors of the tagged player is bounded by

$$f_1(X_t, \xi_t) \;\leq\; N^{-1}\,(2M+1)^{d-1}(M - L + 1)$$

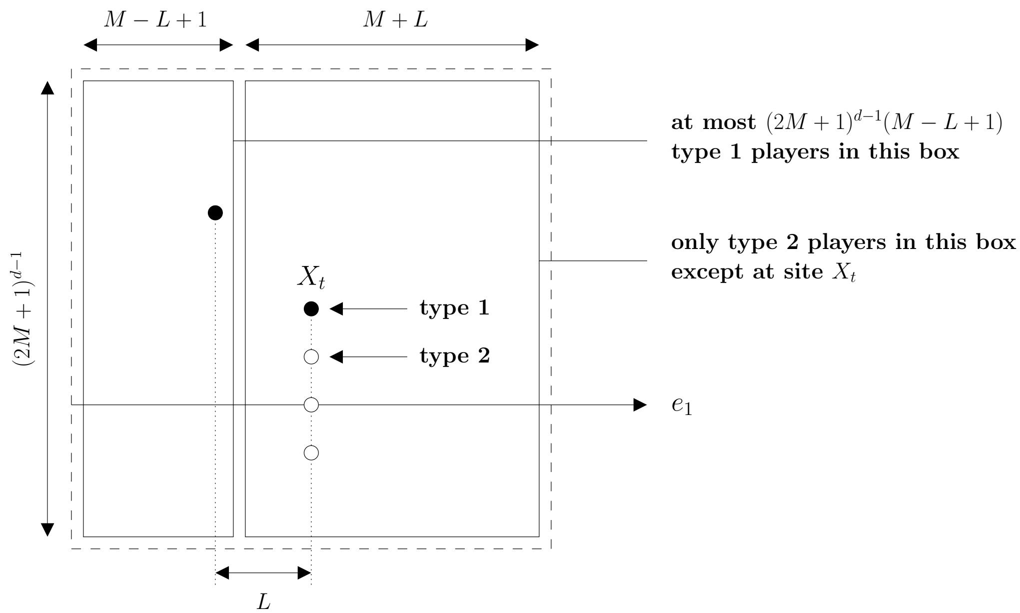


FIGURE 4. Picture related to the proof of Lemma 11.

therefore the rate at which the strategy at the corresponding site changes from $1 \to 2$ which causes the tagged player to jump to the left is

$$\begin{array}{rcl} c(X_t, \xi_t) & \leq & N^{-2}\,(-a_{11})\,\max\,\{z(N-z) : z \leq (2M+1)^{d-1}(M-L+1)\} \\ & = & N^{-2}\,(-a_{11})\,(2M+1)^{d-1}(M-L+1)\,((2M+1)^{d-1}(M+L)-1). \end{array}$$

In addition, each site x occupied by a player of type 2 in the neighborhood of X_t has at least one neighbor of type 1, namely the tagged player, from which it follows that

$$\begin{array}{rcl} c(x, \xi_t) & \geq & N^{-2}\,(-a_{22})\,\min\,\{z(N-z) : z \neq 0\} \\ & = & N^{-2}\,(-a_{22})\,(N-1) \ = \ N^{-2}\,(-a_{22})\,((2M+1)^d - 2). \end{array}$$

Summing over all sites in the right half of the neighborhood of X_t, we get

$$\begin{array}{rcl} \mathcal{D}_t & \geq & N^{-2}\,(-L)(-a_{11})\,(2M+1)^{d-1}(M-L+1)\,((2M+1)^{d-1}(M+L)-1) \\ & & \qquad + \ N^{-2}\,(2M+1)^{d-1}\ \displaystyle\sum_{j=1}^{M}\ j\,(-a_{22})\,((2M+1)^d - 2) \\ & = & N^{-2}\,(2M+1)^{d-1}\ \Big[a_{11}\,L\,(M-L+1)\,((2M+1)^{d-1}(M+L)-1) \\ & & \qquad\qquad - \ a_{22}\ (M(M+1)/2)((2M+1)^d - 2)\Big]. \end{array}$$

Using that $L\,(M-L+1) \leq ((M+1)/2)^2$ and $M+L \leq 2M$, we obtain

$$\begin{array}{rcl} \mathcal{D}_t & \geq & N^{-2}\,(2M+1)^{d-1}\ \Big[a_{11}\,((M+1)/2)^2\,(2M\,(2M+1)^{d-1}-1) \\ & & \qquad\qquad - \ a_{22}\ (M(M+1)/2)((2M+1)^d - 2)\Big] \\ & = & N^{-2}\,(2M+1)^{d-1}((M+1)/4) \\ & & \quad \times\ (a_{11}\ (M+1)(2M\,(2M+1)^{d-1}-1) - a_{22}\ (2M)((2M+1)^d - 2)) \end{array}$$

which is positive whenever

$$a_{11} \ > \ \frac{2M\,((2M+1)^d - 2)}{(M+1)\,(2M\,(2M+1)^{d-1} - 1)} \ a_{22} \ := \ c(M,d)\ a_{22}.$$

In the case when $L_t = L \notin [1, M]$, we have the following alternative:

- $L = 0$ and then there are at least two vertices in the set R_t.
- $L > M$ and then the tagged player has only type 2 players in her neighborhood and therefore changes her strategy at rate zero.

In either case, the process X_t cannot jump to the left whereas it jumps to the right at a positive rate, which again gives the positivity of the drift. This completes the proof. $\square$

The previous lemma is similar to pages 1247–1248 in [15]. There, the authors conclude that we can bring a particle – the tagged player in our case – close to the target Ke_1. This is obvious in one dimension. In higher dimensions, the idea is to use the lemma to increase the first coordinate of the tagged player up to K and then apply the lemma again along each of the other $d-1$ axes to bring the tagged player close to the target. However, since we do not have control on the position of the tagged player in the direction orthogonal to e_1 while moving along the first axis, the conclusion is not obvious. To prove that we can bring a type 1 player close to the target in higher dimensions, we look instead at the Euclidean distance between the target and the type 1 player the closest to the target. We now call X_t the position of one of the type 1 players chosen uniformly at random among the ones who are the closest to Ke_1, that we again call the tagged player, and define

$$\mathfrak{D}_t \ := \ \lim_{h \to 0}\ h^{-1}\, E\,(D_{t+h} - D_t \,|\, \xi_t) \quad \text{where} \quad D_t := \operatorname{dist}(X_t, Ke_1)$$

denotes the Euclidean distance between X_t and the target Ke_1. For our purpose, we only need to prove that the drift $\mathfrak{D}_t$ is almost surely negative when the tagged player is far enough from the target, more precisely on the set of configurations

$$\Omega_K \ := \ \{\eta \subset \mathbb{Z}^d : \eta \cap B_2(Ke_1, K/5) = \varnothing\} \quad \text{for } K \text{ large.}$$

Although our proof relies on basic trigonometry, the algebra is somewhat messy so we only prove the result in the two-dimensional nearest neighbor case. Hopefully, the next lemma will convince the reader that, even if the players are located on a square lattice, the type 1 players spread not only along each axis but also along any arbitrary direction provided condition (25) holds.

Lemma 12 – *Assume that $(M,d) = (1,2)$ and (25) holds. Then,*

$$\mathfrak{D}_t < 0 \quad \textit{almost surely on the event that } \xi_t \in \Omega_K \textit{ for all } K \textit{ large.}$$

Proof. The proof is based on the construction given in Figure 5. Let

$$\begin{array}{rcl} \mathcal{C} & := & \text{the circle with center } Ke_1 \text{ going through } X_t \\ \Delta & := & \text{the tangent line to the circle } \mathcal{C} \text{ going through } X_t \\ \Gamma & := & \text{the straight line parallel to the tangent } \Delta \text{ going through } Ke_1. \end{array}$$

The first ingredient is to observe that, on the event that $\xi_t \in \Omega_K$,

$$\mathfrak{D}_t \ \approx \ \mathfrak{D}_t' \ := \ \lim_{h \to 0}\ h^{-1}\, E\,(\operatorname{dist}(X_{t+h}, \Gamma) - \operatorname{dist}(X_t, \Gamma) \,|\, \xi_t) \tag{28}$$

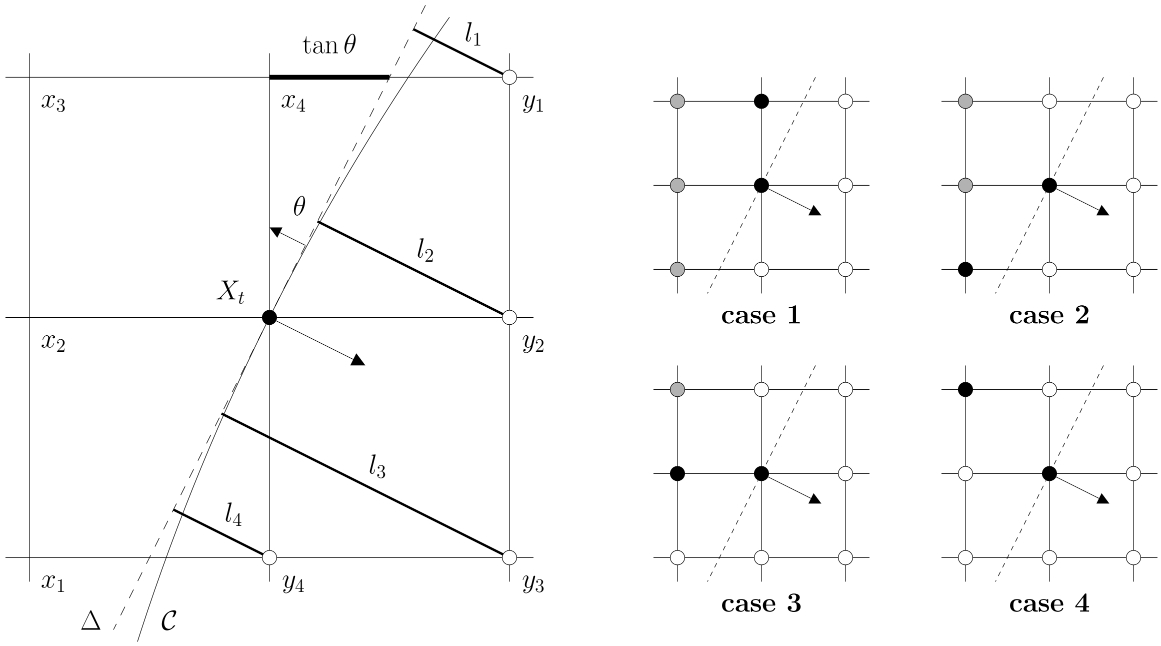


FIGURE 5. Picture related to the proof of Lemma 12.

when the parameter K is large, so it suffices to prove the result for $\mathfrak{D}'_t$. To estimate the drift, note that the straight line Δ divides the neighborhood of X_t into two sets of four vertices. As indicated on the left-hand side of the figure, we denote by y_i the four vertices the closest to the target Ke_1 and we denote by x_i the other four vertices in such a way that

$$l_i \ := \ \mathrm{dist}\,(y_i, \Delta) \ = \ \mathrm{dist}\,(x_i, \Delta) \qquad \text{for } i = 1, 2, 3, 4.$$

Defining the angle θ as in the picture, some basic trigonometry shows that

$$\begin{array}{rclcrcl} l_1 & = & (1 - \tan\theta)\,\cos\theta & \qquad & l_2 & = & \cos\theta \\ l_3 & = & (1 + \tan\theta)\,\cos\theta & \qquad & l_4 & = & \tan\theta\,\cos\theta. \end{array} \tag{29}$$

We may assume that $0 \leq \theta \leq \pi/4$ and so $\tan\theta \in [0, 1]$ since any other configuration can be deduced from a rotation of this configuration. Note that all four players at sites y_i must follow strategy 2, which gives $2^4 = 16$ possible configurations in the neighborhood of X_t. To find a bound for the drift, we only distinguish four types of configurations (see figure).

- **Case 1** – Assume that $x_4 \in \xi_t$. Then,

$$\mathfrak{D}'_t \ \leq \ (-a_{11})\,\frac{4 \times 4}{8 \times 8}\,l_4 - (-a_{22})\left(\frac{2 \times 6}{8 \times 8}\,l_1 \ + \ \frac{2 \times 6}{8 \times 8}\,l_2 \ + \ \frac{1 \times 7}{8 \times 8}\,l_3 \ + \ \frac{1 \times 7}{8 \times 8}\,l_4\right).$$

  Using (29), we deduce that

$$\mathfrak{D}'_t \ \leq \ (1/64)\,\cos\theta\,\left(-a_{11}\,(16\,\tan\theta) \ + \ a_{22}\,(31 + 2\,\tan\theta)\right) \ < \ 0$$

  whenever $a_{11} > (33/16)\,a_{22}$ which holds if $a_{11} > (7/5)\,a_{22}$.

- **Case 2** – Assume that $x_4 \notin \xi_t$ and $x_1 \in \xi_t$. Then,

$$\begin{array}{rcl} \mathfrak{D}_t' & \leq & (1/64)\,\cos\theta\,(-a_{11} \times 15\,l_1 + a_{22} \times (7\,l_1 + 7\,l_2 + 7\,l_3 + 12\,l_4)) \\ & = & (1/64)\,\cos\theta\,(-a_{11}\,(15 - 15\,\tan\theta) + a_{22}\,(21 + 12\,\tan\theta)) \ < \ 0 \end{array}$$

  whenever $a_{11} > (7/5)\,a_{22}$.

- **Case 3** – Assume that $x_4, x_1 \notin \xi_t$ and $x_2 \in \xi_t$. Then,

$$\begin{array}{rcl} \mathfrak{D}_t' & \leq & (1/64)\,\cos\theta\,(-a_{11} \times 12\,l_2 + a_{22} \times (7\,l_1 + 7\,l_2 + 7\,l_3 + 12\,l_4)) \\ & = & (1/64)\,\cos\theta\,(-12\,a_{11} + a_{22}\,(21 + 12\,\tan\theta)) \ < \ 0 \end{array}$$

  whenever $a_{11} > (7/4)\,a_{22}$ which holds if $a_{11} > (7/5)\,a_{22}$.

- **Case 4** – Assume that $x_4, x_1, x_2 \notin \xi_t$ and $x_3 \in \xi_t$. Then,

$$\begin{array}{rcl} \mathfrak{D}_t' & \leq & (1/64)\,\cos\theta\,(-a_{11} \times 7\,l_2 + a_{22} \times (7\,l_1 + 7\,l_2 + 7\,l_3 + 7\,l_4)) \\ & = & (1/64)\,\cos\theta\,(-a_{11}\,(7 + 7\,\tan\theta) + a_{22}\,(21 + 7\,\tan\theta)) \ < \ 0 \end{array}$$

  whenever $a_{11} > 2a_{22}$ which holds if $a_{11} > (7/5)\,a_{22}$.

Since $c(1,2) = 7/5$, we deduce that $\mathfrak{D}_t' < 0$ whenever (25) holds, which gives the desired result according to the approximation (28) when the parameter K is large. □

We now use Lemmas 11 and 12 to prove that, with probability close to one when K is large, the tagged player is located in a certain Euclidean ball with center Ke_1 at a deterministic time proportional to K. This is done in Lemmas 13–15 below where we successively prove that the tagged player hits a subset of the target region in a short time, does not leave a certain larger ball centered at zero, and stays in the target region for a long time. The second step is needed to ensure that the events under consideration are measurable with respect to the graphical representation in a finite space-time box, which is a key to obtaining a coupling between the process and oriented percolation with a finite range of dependence. For every positive integer K, define

$$\tau_K \ := \ \inf\{t : X_t \in B_2(Ke_1, 2K/5)\} \ = \ \inf\{t : D_t < 2K/5\}.$$

Lemma 13 – *Assume* (25). *There exist $c, C_1 < \infty$ and $\gamma_1 > 0$ such that*

$$P\,(\tau_K \geq cK \,|\, X_0 \in B_2(0, 3K/5)) \ \leq \ C_1\,\exp(-\gamma_1 K) \quad \textit{for all } K \textit{ large.}$$

Proof. According to Lemmas 11 and 12,

$$\lim_{h \to 0} h^{-1}\,E\,(D_{t+h} - D_t \,|\, D_t \geq K/5) \ \leq \ -\mu \quad \text{for some} \quad \mu > 0 \tag{30}$$

from which it follows that

$$E\,D_t \ \leq \ D_0 - \mu t \quad \text{for all} \quad t < \mu^{-1}(D_0 - K/5).$$

Large deviation estimates for the Poisson distribution then imply that

$$\begin{array}{l} P\,(\tau_K \geq 2\mu^{-1}K \,|\, X_0 \in B_2(0, 3K/5)) \\ \qquad \leq \ P\,(D_t \geq K/5 \text{ for all } t < 2\mu^{-1}K \,|\, D_0 \leq 7K/5) \ \leq \ C_1\,\exp(-\gamma_1 K) \end{array}$$

for suitable constants $C_1 < \infty$ and $\gamma_1 > 0$, and all K sufficiently large. □

Lemma 14 – *Assume* (25). *There exist* $C_2 < \infty$ *and* $\gamma_2 > 0$ *such that*

$$P\,(D_t \geq 2K \ \textit{for some}\ t \in (0, cK) \,|\, X_0 \in B_2(0, 3K/5)) \ \leq \ C_2 \, \exp(-\gamma_2 K)$$

for all K *sufficiently large and where* c *is as in Lemma 13.*

PROOF. First, we introduce the stopping times

$$\sigma_K \ := \ \inf \{t : D_t \geq 2K\} \quad \text{and} \quad \tau \ := \ \inf\,(\tau_K, \sigma_K)$$

and for all $a > 0$ the process stopped at time τ

$$\phi_t(a) \ := \ a^{D_t} \ \mathbf{1}\{t < \tau\} + a^{D_\tau} \ \mathbf{1}\{t \geq \tau\}.$$

The key to the proof is to find $a > 1$ such that the process $\phi_t(a)$ is a supermartingale with respect to the natural filtration of the process ξ_t and then apply the optimal stopping theorem. To prove the existence of such a constant, we introduce

$$\psi_t(a) \ := \ \lim\nolimits_{h \to 0} \ h^{-1} \, E\,(\phi_{t+h}(a) - \phi_t(a) \,|\, \xi_t)$$

and observe that

$$\psi_t(a) \ = \ \sum_{x \in \mathbb{Z}^d} \Big(a^{\,\mathrm{dist}\,(X_t + x, Ke_1)} - a^{\,\mathrm{dist}\,(X_t, Ke_1)}\Big) \ \lim_{h \to 0} \ P\,(X_{t+h} - X_t = x \,|\, \xi_t).$$

Recalling (30) and using that $D_t \geq K/5$ for all $t < \tau$, we deduce that

$$\begin{aligned} \psi_t'(1) \ &= \ \sum_{x \in \mathbb{Z}^d} \Big(\mathrm{dist}\,(X_t + x, Ke_1) - \mathrm{dist}\,(X_t, Ke_1)\Big) \ \lim_{h \to 0} \ P\,(X_{t+h} - X_t = x \,|\, \xi_t) \\ &= \ \lim_{h \to 0} \ h^{-1} \, E\,(D_{t+h} - D_t \,|\, \xi_t) \ \leq \ -\mu \quad \text{almost surely.} \end{aligned}$$

Since in addition $\psi_t(1) = 0$ almost surely, there exists $a > 1$ such that

$$\psi_t(a) \ := \ \lim_{h \to 0} \ h^{-1} \, E\,(\phi_{t+h}(a) - \phi_t(a) \,|\, \xi_t) \ \leq \ 0$$

which implies that $\phi_t(a)$ is a supermartingale. Since the stopping time τ is almost surely finite, the optimal stopping theorem further implies that

$$\begin{aligned} E\,\phi_\tau(a) \ &\leq \ E\,\phi_0(a) \ \leq \ a^{K + 3K/5} \ = \ a^{8K/5} \\ E\,\phi_\tau(a) \ &\geq \ a^{2K} \ P\,(\sigma_K < \tau_K) \ + \ a^{2K/5 - M} \ P\,(\tau_K < \sigma_K) \\ &\geq \ a^{2K} \ P\,(\sigma_K < \tau_K) \ + \ a^{2K/5 - M} \ (1 - P\,(\sigma_K < \tau_K)) \end{aligned}$$

from which we deduce that

$$\begin{aligned} P\,(\sigma_K < \tau_K) \ &\leq \ (a^{8K/5} - a^{2K/5 - M})(a^{2K} - a^{2K/5 - M})^{-1} \\ &= \ (a^{6K/5 + M} - 1)(a^{8K/5 + M} - 1)^{-1} \ \leq \ a^{-2K/5}. \end{aligned}$$

Since the probability that the number of jumps by time cK exceeds a certain multiple of K also has exponential decay, the result follows. $\square$

Lemma 15 – *Assume* (25). *There exist* $C_3 < \infty$ *and* $\gamma_3 > 0$ *such that*

$$P\,(D_t \geq 3K/5 \text{ for some } t \in (\tau_K, cK) \,|\, \tau_K < cK) \ \leq \ C_3 \, \exp(-\gamma_3 K)$$

for all K *sufficiently large.*

Proof. The result directly follows by observing that

$$\begin{aligned} & P\,(D_t \geq 3K/5 \text{ for some } t \in (\tau_K, cK) \,|\, \tau_K < cK) \\ & \qquad \leq \ P\,(D_t \geq 3K/5 \text{ for some } t \in (0, cK) \,|\, D_0 < 2K/5) \end{aligned}$$

and by following the argument of the proof of Lemma 14 but using

$$\tau_K' \ := \ \inf\,\{t : D_t < K/5\} \quad \text{and} \quad \sigma_K' \ := \ \inf\,\{t : D_t \geq 3K/5\}$$

in place of the stopping times τ_K and σ_K. $\square$

With Lemmas 13–15, we are now ready to couple the process properly rescaled in space and time with oriented site percolation. Denote by $\mathbb{W}_n^{1-\epsilon}$ the set of wet sites at level n in a one dependent oriented site percolation process on $\mathcal{H}_1$ in which sites are open with probability $1-\epsilon$.

Lemma 16 – *Assume* (25) *and let* $\epsilon > 0$. *Then, for all* K *sufficiently large, the process can be coupled with oriented site percolation in such a way that*

$$\mathbb{W}_n^{1-\epsilon} \subset \mathbb{X}_n \text{ for all } n \quad \text{whenever} \quad \mathbb{X}_0 = \mathbb{W}_0^{1-\epsilon}.$$

Proof. Let $\Omega(z,n)$ denote the event that site $(z,n) \in H$ is occupied, which has been defined in (26). Lemmas 13 and 15 implies the existence of a collection of events $G(z,n)$ measurable with respect to the graphical representation of the process such that

1. for all K sufficiently large, $P\,(G(z,n)) \geq 1-\epsilon$, and such that
2. we have the inclusions of events

$$G(z,n) \,\cap\, \Omega(z,n) \,\subset\, \Omega(z \pm e_i, n+1) \quad \text{for all} \quad i = 1, 2, \ldots, d.$$

In addition, Lemma 14 implies that these events can be chosen so that

3. $G(z,n)$ is measurable with respect to the graphical representation in

$$B_2(Kz, 2K) \times [cnK, c(n+1)K] \ = \ (Kz, cnK) + B_2(0, 2K) \times [0, cK].$$

These are the assumptions of Theorem 4.3 in Durrett [6], from which the existence of the coupling between the two processes directly follows. $\square$

In the next lemma, which recalls the statement of Theorem 5, we return to the process with general payoffs. The proof relies on the previous lemma, the symmetry of the evolution rules of the spatial game and a perturbation argument.

Lemma 17 – *For all* a_{12} *and* a_{21} *there exists* $m > 0$ *such that coexistence occurs when*

$$c(M,d)\, a_{22} < a_{11} < -m \quad \text{and} \quad c(M,d)\, a_{11} < a_{22} < -m.$$

PROOF. First, we fix $\epsilon < (1/2)(1-p_c)$ positive where $p_c < 1$ is the critical value of the oriented site percolation process introduced above. To prove that both strategies can survive simultaneously, we extend our previous definition of occupied site by calling $(z,n) \in H$ a **good** site whenever

$$x \in \eta_{cnK} \quad \text{and} \quad y \notin \eta_{cnK} \quad \text{for some} \quad x, y \in B_2(Kz, 3K/5).$$

Denote by $\mathbb{Y}_n$ the set of good sites at level n. The symmetry of the evolution rules implies that the conclusion of Lemma 16 holds for $\mathbb{Y}_n$ provided that

$$a_{12} = a_{21} = 0 \quad \text{and} \quad c(M,d)\, a_{22} < a_{11} < -1 \quad \text{and} \quad c(M,d)\, a_{11} < a_{22} < -1. \tag{31}$$

Even though (weak) survival of both strategies when $a_{12} = a_{21} = 0$ is in fact trivial since in this case a player isolated from players of her own type cannot change her strategy, we point out that the coupling with oriented site percolation is needed to obtain the full coexistence region. Indeed, the parameter K being fixed such that the process dominates one dependent oriented site percolation with parameter $1-\epsilon$, the continuity of the transition rates with respect to the payoffs implies the existence of a small $\rho = \rho(K) > 0$ and a coupling of the processes such that

$$\mathbb{W}_n^{1-2\epsilon} \subset \mathbb{Y}_n \text{ for all } n > 0 \quad \text{whenever} \quad \mathbb{Y}_0 = \mathbb{W}_0^{1-2\epsilon} \tag{32}$$

in a perturbation of the parameter region (31) given by

$$-\rho < a_{12}, a_{21} < \rho \quad \text{and} \quad c(M,d)\, a_{22} < a_{11} < -1 \quad \text{and} \quad c(M,d)\, a_{11} < a_{22} < -1. \tag{33}$$

In particular, letting $f : \{0,1\}^H \to \{0,1\}$ be defined by

$$f(\{\mathbb{W}_n : n \geq 0\}) \ := \ \mathbf{1}\{\text{card}\,(n : z \in \mathbb{W}_n) = \infty \text{ for all } z \in \mathbb{Z}^d\} \tag{34}$$

and using (32) and the monotonicity of f, we obtain that, for all $(x,t) \in \mathbb{Z}^d \times \mathbb{R}_+$

$$\begin{aligned} & P\,(x \in \eta_{s_1} \text{ and } x \notin \eta_{s_2} \text{ for some } s_1, s_2 > t) \\ & \quad \geq \ P\,(\text{card}\,(n : z \in \mathbb{Y}_n) = \infty \text{ for all } z \in \mathbb{Z}^d) \ = \ E\, f(\{\mathbb{Y}_n : n \geq 0\}) \\ & \quad \geq \ E\, f(\{\mathbb{W}_n^{1-2\epsilon} : n \geq 0\}) \ = \ P\,(\text{card}\,(n : z \in \mathbb{W}_n^{1-2\epsilon}) = \infty \text{ for all } z \in \mathbb{Z}^d) \ = \ 1 \end{aligned}$$

since infinitely many sites are wet at level zero and $1 - 2\epsilon > p_c$. This proves coexistence of both strategies in the parameter region (33). To deal with the general case when both payoffs a_{12} and a_{21} are arbitrary, we fix a sufficiently large $m > 0$ such that

$$a_{12} \in (-m\rho, m\rho) \quad \text{and} \quad a_{21} \in (-m\rho, m\rho).$$

Since the long-term behavior remains unchanged by speeding up time by m, i.e., multiplying all the payoffs by the same factor m, we obtain coexistence in the parameter region

$$c(M,d)\, a_{22} < a_{11} < -m \quad \text{and} \quad c(M,d)\, a_{11} < a_{22} < -m.$$

This proves the lemma and Theorem 5. $\square$

5. Proof of Theorem 2

This section is devoted to the proof of Theorem 2. We first prove that, under the assumption of the theorem, strategy 1 survives. The key to obtaining this partial result is to observe that, when the first payoff $a_{11} = 1$ while the other payoffs are equal to zero, the spatial game starting from suitable initial configurations dominates stochastically a Richardson model [18]. To also prove extinction of strategy 2, we use an idea of the author [10] that extends from discrete-time to continuous-time processes a result of Durrett [5] which states that sites which are not wet do not percolate for oriented site percolation models in which sites are open with probability close to one. Throughout this section, to shorten a little bit the expression of certain events, we let

$$B_r \ := \ [-r, r]^d \quad \text{for all} \quad r > 0.$$

The spatial boxes involved in the block construction in both this section and the next section are translations of these boxes for an appropriate radius r.

Lemma 18 – *Let $\epsilon > 0$ and $a_{11} = 1$. Then, there exist $K, c, \rho > 0$ such that*

$$P\,(\eta_t \not\supset B_{2K} \ \textit{for some} \ t \in (cK, 2cK) \,|\, \eta_0 \supset B_K) \ \leq \ \epsilon \quad \textit{for all} \quad \bar a_{11} \in (-\rho, \rho)^3.$$

Proof. We introduce the following auxiliary processes:

- the spatial game ξ_t with payoffs $a_{11} = 1$ and $\bar a_{11} = 0$, and
- the Richardson model ζ_t with flip rate

$$c_{RM}(x, \zeta) \ = \ \beta^2 \, f_1(x, \zeta) \, \mathbf{1}\{x \notin \zeta\} \quad \text{where} \quad \beta := ((2M+1)^d - 1)^{-1}.$$

In the process ξ_t all type 2 players have a zero payoff while each type 1 player with at least one type 1 neighbor has a payoff equal to β from which it follows that

$$\begin{array}{rcll} c_{SG}(x, \xi) & = & 0 & \text{if} \quad x \in \xi \\ c_{SG}(x, \xi) & \geq & \beta^2 & \text{if} \quad x \notin \xi \ \text{and} \ f_1(y, \xi) \neq 0 \ \text{for some} \ y \in \xi \cap N_x. \end{array}$$

Since in addition the property that each type 1 player has at least one type 1 neighbor is preserved by the dynamics of the processes, we deduce the existence of a coupling (ζ, ξ) such that

$$P\,(\zeta_t \subset \xi_t \mid \zeta_0 = \xi_0 = B_K) \ = \ 1. \tag{35}$$

In other respects, the shape theorem [18] for the Richardson model implies the existence of a positive constant $c > 0$ fixed from now on such that

$$\begin{array}{l} P\,(\zeta_t \not\supset B_{2K} \ \text{for some} \ t \in (cK, 2cK) \mid \zeta_0 \supset B_K) \\ \quad = \ P\,(\zeta_{cK} \not\supset B_{2K} \mid \zeta_0 \supset B_K) \\ \quad \leq \ P\,(\zeta_{cK} \not\supset B_{2K} \mid \zeta_0 = \{0\}) \ \leq \ \epsilon/2 \quad \text{for all } K \text{ large} \end{array}$$

where the equality between the first two lines holds because infected sites in the Richardson model do not recover. In view of (35), the same holds for the spatial game, i.e.,

$$P\,(\xi_t \not\supset B_{2K} \ \text{for some} \ t \in (cK, 2cK) \,|\, \xi_0 \supset B_K) \ \leq \ \epsilon/2 \tag{36}$$

for all K large. Now, we fix K such that (36) holds. The scale parameter K and the constant c being fixed, the continuity of the transition rates of the spatial game with respect to the payoffs implies the existence of a small constant $\rho > 0$ and a coupling (η, ξ) such that

$$P\,(\eta_t \cap B_{2K} \neq \xi_t \cap B_{2K} \text{ for some } t \leq 2cK \mid \eta_0 = \xi_0) \ \leq \ \epsilon/2 \tag{37}$$

whenever $\bar a_{11} \in (-\rho, \rho)^3$. Combining (36) and (37) gives

$$\begin{aligned} & P\,(\eta_t \not\supset B_{2K} \text{ for some } t \in (cK, 2cK) \,|\, \eta_0 \supset B_K) \\ & \quad \leq \ P\,(\xi_t \not\supset B_{2K} \text{ for some } t \in (cK, 2cK) \,|\, \xi_0 \supset B_K) \\ & \qquad + \ P\,(\eta_t \cap B_{2K} \neq \xi_t \cap B_{2K} \text{ for some } t \leq 2cK \mid \eta_0 = \xi_0) \ \leq \ \epsilon \end{aligned}$$

for all $a_{11} \in (-\rho, \rho)^3$. This completes the proof. $\square$

To deduce survival of strategy 1 from Lemma 18 and under the assumptions of the lemma, we now declare a site $(z, n) \in H$ to be **occupied** whenever

$$x \in \eta_t \quad \text{for all} \quad (x, t) \in (Kz, cnK) + B_K \times (0, cK)$$

and define the set $\mathbb{X}_n$ of occupied sites at level n as in (27). Repeating the proof of Lemma 16 but using Lemma 18 in place of Lemmas 13–15 directly gives the following result.

Lemma 19 – *Let $\epsilon > 0$ and $a_{11} = 1$. Then, there exist $K, c, \rho > 0$ and a coupling of the spatial game with one dependent oriented site percolation such that*

$$\mathbb{W}_n^{1-\epsilon} \subset \mathbb{X}_n \textit{ for all } n \quad \textit{whenever} \quad \mathbb{X}_0 = \mathbb{W}_0^{1-\epsilon} \ \textit{ and } \ \bar a_{11} \in (-\rho, \rho)^3.$$

Taking $\epsilon > 0$ strictly smaller than one minus the critical value of one dependent oriented site percolation, and using the coupling given in the previous lemma for this value of ϵ as well as the monotone function f defined in (34), we obtain that, for all $(x, t) \in \mathbb{Z}^d \times \mathbb{R}_+$

$$\begin{aligned} & P\,(x \in \eta_s \text{ for some } s > t) \\ & \quad \geq \ P\,(\mathrm{card}\,(n : z \in \mathbb{X}_n) = \infty \text{ for all } z \in \mathbb{Z}^d) \ = \ E\, f(\{\mathbb{X}_n : n \geq 0\}) \\ & \quad \geq \ E\, f(\{\mathbb{W}_n^{1-\epsilon} : n \geq 0\}) \ = \ P\,(\mathrm{card}\,(n : z \in \mathbb{W}_n^{1-\epsilon}) = \infty \text{ for all } z \in \mathbb{Z}^d) \ = \ 1. \end{aligned}$$

This proves that strategy 1 survives but not that it wins since there is a positive density of closed sites, which does not exclude the possibility of having a positive density of sites which are not occupied, and so the presence of type 2 players, at arbitrarily large times. To prove extinction of the type 2 players, we use the coupling above together with an idea of the author [10] that extends a result of Durrett [5]. This is done in the next lemma.

Lemma 20 – *Let $a_{11} = 1$. Then, there exists $\rho > 0$ small such that*

$$\lim_{t \to \infty} \ P\,(x \notin \eta_t) \ = \ 0 \quad \textit{for all } x \in \mathbb{Z}^d \quad \textit{whenever } \bar a_{11} \in (-\rho, \rho)^3.$$

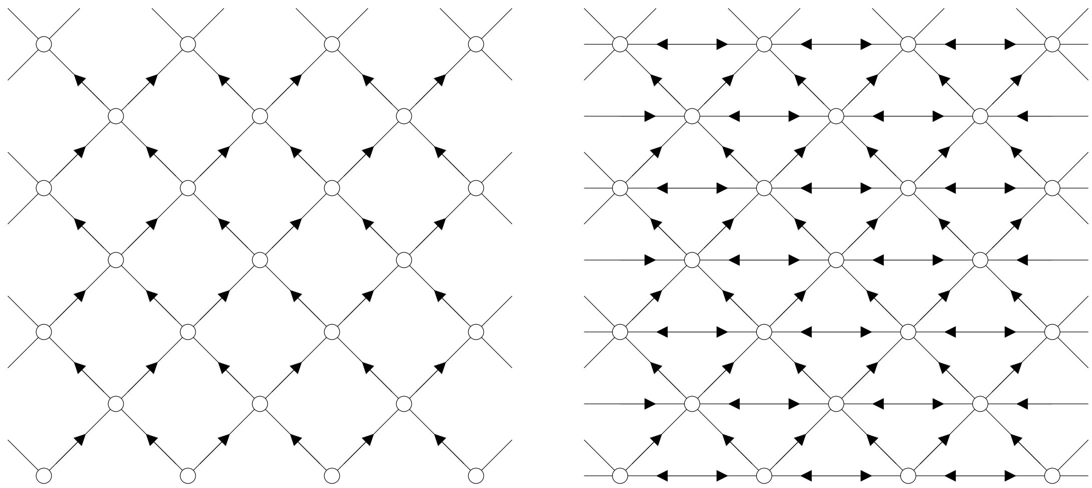


FIGURE 6. Picture of the graphs $\mathcal{H}_1$ and $\mathcal{H}_2$ in dimension $d = 1$.

PROOF. Following an idea of [10] we introduce the new oriented graph $\mathcal{H}_2$ with the same vertex set as the oriented graph $\mathcal{H}_1$ but in which there is an oriented edge

$$\begin{array}{l} (z, n) \to (z', n') \quad \text{if and only if} \\ \qquad (z' = z \pm e_i \text{ for some } i = 1, 2, \ldots, d \ \text{ and } \ n' = n + 1) \\ \qquad\quad \text{or } \ (z' = z \pm 2e_i \text{ for some } i = 1, 2, \ldots, d \ \text{ and } \ n' = n). \end{array}$$

See the right-hand side of Figure 6 for a picture in $d = 1$. We say that a site is dry if it is not wet for oriented site percolation on the graph $\mathcal{H}_1$. Also, we write

$$(w, 0) \ \to_j \ (z, n) \quad \text{for } j = 1, 2$$

and say that there is a dry path in $\mathcal{H}_j$ connecting both sites if there exist

$$(z_0, 0) = (w, 0), \ (z_1, n_1), \ \ldots, (z_{k-1}, n_{k-1}), \ (z_k, n_k) = (z, n) \in H$$

such that the following two conditions hold:

1. $(z_i, n_i) \to (z_{i+1}, n_{i+1})$ is an oriented edge in $\mathcal{H}_j$ for all $i = 0, 1, \ldots, k - 1$ and
2. the site (z_i, n_i) is dry for all $i = 0, 1, \ldots, k$.

Note that a dry path in $\mathcal{H}_1$ is also a dry path in $\mathcal{H}_2$ but the reciprocal is false since the latter has more oriented edges than the former. The key to the proof is the following result: if sites are closed with probability $\epsilon > 0$ sufficiently small then

$$\lim_{n \to \infty} \ P\left((w, 0) \to_2 (z, n) \text{ for some } w \in \mathbb{Z}^d\right) \ = \ 0. \tag{38}$$

In other words, if the density of open sites is close enough to one then dry sites do not percolate even with the additional edges in $\mathcal{H}_2$. The proof for dry paths in the graph $\mathcal{H}_1$ directly follows from Lemmas 4–11 in Durrett [5] but as pointed out in [10], the proof easily extends to give the

analog for dry paths in the oriented graph $\mathcal{H}_2$. To complete the proof, the last step is to show the connection between dry paths and occupied sites. Assume that

$$x \notin \eta_t \quad \text{for some} \quad (x,t) \in (Kz, cnK) + B_K \times (0, cK) \tag{39}$$

where $(z,n) \in H$. Since a type 1 player can only change her strategy if there is a type 2 player in her neighborhood, the event in (39) implies the existence of

$$x_0, x_1, \ldots, x_m = x \in \mathbb{Z}^d \quad \text{and} \quad s_0 = 0 < s_1 < \cdots < s_{m+1} = t$$

such that the following two conditions hold:

1. for all $j = 0, 1, \ldots, m$, we have $x_j \notin \eta_s$ for $s \in [s_j, s_{j+1}]$ and
2. for all $j = 0, 1, \ldots, m-1$, we have $x_{j+1} \in N_{x_j}$.

In particular, the spatial game being coupled with one dependent oriented site percolation as in Lemma 19, the event in (39) implies that there exists a dry path

$$(w,0) \to_2 (z,n) \quad \text{for some} \quad w \in \mathbb{Z}^d \tag{40}$$

provided that the range $M \leq K$. Note however that this does not imply the existence of a dry path in the graph $\mathcal{H}_1$ which is the reason why we introduced a new graph with additional edges. In conclusion, the event in (39) is a subset of the event (40). Since in addition ϵ can be made arbitrarily small by choosing the parameter K sufficiently large according to Lemma 19, taking the probability of the two events above and using (38) implies that

$$\lim_{t\to\infty} P\,(x \notin \eta_t) \leq \lim_{n\to\infty} P\,((w,0) \to_2 (z,n) \text{ for some } w \in \mathbb{Z}^d) \;=\; 0$$

for all $x \in \mathbb{Z}^d$ where (z,n) is as in (39). This completes the proof. $\square$

To complete the proof of the theorem, we let $\rho > 0$ as in the previous lemma. Then, having arbitrary payoffs a_{12} and a_{21} and a_{22}, there exists m large such that $\bar{a}_{11} \in (-m\rho, m\rho)$. Since in addition the limiting behavior of the spatial game remains unchanged by speeding up time, Lemma 20 implies that strategy 1 wins for all $a_{11} > m$.

6. Proof of Theorem 3

This section is devoted to the proof of Theorem 3. The intuition behind this result is simple, though the arguments to make the proof rigorous are somewhat more challenging. To understand the theorem heuristically, observe that, in the limiting case $a_{11} = -\infty$ and provided one starts from a suitable initial configuration, the process becomes instantaneously sparse: configurations where two type 1 players are neighbors are not possible. Since in addition type 2 players can only change their strategy when they are located in the neighborhood of at least one type 1 player, the process is dominated by a system of annihilating particles: as long as several particles are in the same interaction neighborhood, one of them is instantaneously killed. In particular, the density of type 1 players can only decrease. Under the assumption $a_{12} < 0$, these particles also die spontaneously, which implies that the density of type 1 players decreases to zero.

The main difficulty to prove the theorem is to extend this heuristic argument to the non-limiting case when the payoff a_{11} is small but different from $-\infty$. We start with some key definitions and a brief overview of the global strategy of our proof. Identifying again configurations with the set of type 1 players, we say that a set/configuration η is **sparse** whenever

$$x, y \in \eta \quad \text{implies that} \quad y \notin N_x.$$

We also say that configuration η is sparse in B if the set $\eta \cap B$ is sparse. We say that there is a **type 1 invasion path** $(x, r) \rightsquigarrow (y, t)$ if there are

$$x_0 = x, x_1, \ldots, x_n = y \in \mathbb{Z}^d \quad \text{and} \quad s_0 = r < s_1 < \cdots < s_n < s_{n+1} = t \in \mathbb{R}_+$$

such that the following three conditions hold:

- for $i = 0, 1, \ldots, n$, we have $x_i \in \eta_s$ for all $s_i \leq s \leq s_{i+1}$
- for $i = 1, 2, \ldots, n$, we have $x_i \notin \eta_{s_i -}$
- for $i = 1, 2, \ldots, n$, we have $x_i \in N_{x_{i-1}}$.

Note that there exists a type 1 invasion path $(x, 0) \rightsquigarrow (y, t)$ if and only if the player at site y at time t follows strategy 1 since type 2 players can only change their strategy if they are in the neighborhood of a type 1 player. Finally, we call an invasion path

- an **inner path** whenever $x_i \in B_{4K-M}$ for all $i = 0, 1, \ldots, n$,
- an **outer path** whenever $x_i \notin B_{4K}$ for all $i = 0, 1, \ldots, n$,
- a **transversal path** whenever $x_i \in B_{4K} \setminus B_{4K-M}$ for some $i = 0, 1, \ldots, n$.

To prove extinction of strategy 1 using a block construction, the main ingredient is to prove that if the region B_K is empty initially then the region B_{2K} will, with probability close to one for suitable parameters, be empty at a later time that we choose to be $2\sqrt{K}$. To show this result, we observe that, since type 1 players at located on type 1 invasion paths, it suffices to prove that

1. the probability that an inner path lasts more than $2\sqrt{K}$ units of time is small and
2. the probability that a transversal path reaches B_{2K} by time $2\sqrt{K}$ is small.

Note that outer paths are unimportant in proving the theorem because, by definition, they do not reach the target region. The proof of the second assertion simply relies on the fact that, with probability close to one when K is large and regardless of the value of the payoffs, invasion paths expand at most linearly. The proof of the first assertion is more involved and is divided into three steps. First, we show that the process is sparse in B_{4K} by time 1 then that it is sparse a positive fraction of time in this box until time $2\sqrt{K}$, and finally that, conditional on this previous event, inner paths die out exponentially fast. Throughout this section,

- $\mathcal{S}$ denotes the set of sparse configurations,
- Ω_{in} is the event that there is an inner path from time 0 to time $2\sqrt{K}$ and
- Ω_{tr} is the event that a transversal path reaches B_{2K} by time $2\sqrt{K}$.

To begin with, we prove that if the region B_K is initially void in type 1 players then the process becomes sparse in B_{4K} after a short time.

Lemma 21 – *Let $\epsilon > 0$. For all K, there exists $m_1 := m_1(\epsilon, K, \bar a_{11}) < \infty$ such that*

$$P\left(\eta_s \cap B_{4K} \notin \mathcal{S} \text{ for all } s \in (0,1) \,|\, \eta_0 \cap B_K = \varnothing\right) \leq \epsilon/3 \quad \text{for all} \quad a_{11} < -m_1.$$

Proof. To begin with, we observe that when $a_{11} = -1$ and $\bar a_{11} = 0$,

- type 1 players with at least one type 1 neighbor and at least one type 2 neighbor change their strategy at a positive rate whereas
- type 2 players all have a zero payoff so they do not change their strategy.

This implies that there exists $a := a(\epsilon) > 0$ such that, for all $K > 0$,

$$P\left(\eta_{aK} \cap B_{4K} \notin \mathcal{S} \mid \eta_0 \cap B_K = \varnothing\right) \leq \epsilon/6 \quad \text{when} \quad \bar a_{11} = 0$$

which in turn implies the existence of $\rho := \rho(K, \epsilon)$ such that

$$P\left(\eta_{aK} \cap B_{4K} \notin \mathcal{S} \mid \eta_0 \cap B_K = \varnothing\right) \leq \epsilon/3 \quad \text{for all} \quad \bar a_{11} \in (-\rho, \rho)^3. \tag{41}$$

For arbitrary $\bar a_{11}$, we fix $m_1 := m_1(\bar a_{11}, \rho)$ such that

$$\bar a_{11} \in (-m_1 \rho, m_1 \rho)^3 \quad \text{and} \quad m_1 > aK.$$

Then, (41) directly implies that

$$P\left(\eta_s \cap B_{4K} \notin \mathcal{S} \mid \eta_0 \cap B_K = \varnothing\right) \leq \epsilon/3$$

for all $a_{11} < -m_1$ and for $s = aK/m_1 < 1$. $\square$

Before proving that inner paths die out exponentially fast, we need an additional preliminary result that ensures that the configuration in B_{4K} is almost sparse for a large amount of time when a_{11} is small. The proof slightly differs depending on the sign of the two payoffs a_{22} and a_{21}. Since the proof when these two payoffs are negative is more difficult and requires additional arguments, we only focus on this case. Under this assumption and the assumptions of the theorem, all four payoffs are negative, in which case all the players have a positive death rate and the graphical representation of the process can be reformulated in the following manner. We introduce the following collections of independent random variables: for all $x \in \mathbb{Z}^d$ and $i, j = 1, 2$ and $n > 0$,

- we let $T_n(x, i, j)$ = the nth arrival time of a Poisson process with rate $-a_{ij}$
- we let $U_n(x, i, j)$ = a uniform random variable over the interaction neighborhood N_x
- we let $V_n(x, i, j)$ = a uniform random variable over the interaction neighborhood N_x.

At the arrival times $T_n(x, i, j)$, we draw

$$\text{an arrow} \quad V_n(x, i, j) \to x \quad \text{with the label} \quad (U_n(x, i, j), i, j) \tag{42}$$

and say that this arrow is **active** whenever

$$\eta_{t-}(x) = i \quad \text{and} \quad \eta_{t-}(U_n(x, i, j)) = j \quad \text{where} \quad t := T_n(x, i, j).$$

Given an initial configuration and a realization of this graphical representation, the process can be constructed going forward in time by setting

$$\begin{array}{rcll} \eta_t(x) & := & \eta_{t-}(V_n(x,i,j)) & \text{if the arrow in (42) is active} \\ \eta_t(x) & := & \eta_{t-}(x) & \text{if the arrow in (42) is not active} \end{array}$$

where again $t := T_n(x,i,j)$. For any given $K > 0$, we let $\tau_0 = 0$ and

$$\begin{array}{rcl} \tau_i & := & \inf\,\{T_n(x,2,j) > \tau_{i-1} : n > 0 \text{ and } x \in B_{4K} \text{ and } j = 1,2\} \\ \rho_i & := & (1/2)(\tau_i + \tau_{i+1}) \quad \text{for } i \geq 1 \end{array}$$

and say that the arrow at time τ_i is **good** whenever

$$\begin{array}{l} \text{there is at least one } (x,1,1)\text{-arrow } z' \to z \\ \qquad\qquad \text{for all } z \in N_x \text{ and all } z' \in N_z \text{ between time } \tau_i \text{ and time } \rho_i. \end{array}$$

Note that if the arrow at time τ_i is good and the configuration in B_{4K} is sparse just before τ_i and the player at x becomes of type 1 at time τ_i then all the type 1 players in the neighborhood of x become of type 2 by time ρ_i unless the player at x changes her strategy before. In either case, the configuration will be sparse in B_{4K} between ρ_i and τ_{i+1}. Define the stopping time

$$\sigma_K \ := \ \inf\,\{\tau_i : \text{the arrow at time } \tau_i \text{ is not good}\}.$$

Then, we have the following lemma.

Lemma 22 – *Let $\epsilon > 0$. For all K, there exists $m_2 := m_2(\epsilon, K, \bar a_{11}) < \infty$ such that*

$$P\,(\sigma_K < 2\sqrt{K}) \ \leq \ \epsilon/6 \quad \textit{for all} \quad a_{11} < -m_2.$$

PROOF. First, we observe that

$$J \ := \ \sup\,\{j : \tau_j < 2\sqrt{K}\} \ = \ \text{Poisson}\,(-2\,(a_{21} + a_{22})\sqrt{K}\,(8K+1)^d)$$

and fix $m := m(\epsilon, K, \bar a_{11})$ such that

$$P\,(J > m) \ \leq \ \epsilon/18. \tag{43}$$

Letting $\rho := \rho(\epsilon, K, \bar a_{11}) = -\epsilon\,(36\,m\,(a_{21} + a_{22})(8K+1)^d)^{-1}$ and using that

$$\tau_{i+1} - \tau_i \ = \ \text{Exponential}\,(-(a_{21} + a_{22})(8K+1)^d)$$

we also have

$$\begin{array}{l} P\,(\tau_{i+1} - \tau_i < 2\rho \text{ for some } i = 0,1,\ldots,J-1 \,|\, J \leq m) \\ \qquad\qquad \leq \ P\,(\tau_{i+1} - \tau_i < 2\rho \text{ for some } i = 0,1,\ldots,m-1) \\ \qquad\qquad \leq \ m\ (1 - \exp(2\,(a_{21} + a_{22})(8K+1)^d\,\rho)) \\ \qquad\qquad \leq \ -2m\,(a_{21} + a_{22})(8K+1)^d\,\rho \ = \ \epsilon/18. \end{array} \tag{44}$$

Finally, since for all $x \in B_{4K}$, all $z \in N_x$ and all $z' \in N_z$,

$$\begin{aligned} & P\,(\text{there is no } (x,1,1)\text{-arrow } z' \to z \text{ between time } \tau_i \text{ and time } \tau_i + \rho) \\ & \qquad = \ P\,(\text{Exponential}\,(-a_{11}\,(2M+1)^{-2d}) > \rho) \ = \ \exp(a_{11}\,(2M+1)^{-2d}\,\rho) \end{aligned}$$

defining $m_2 := m_2(\epsilon, K, \bar a_{11})$ by

$$m_2 \ := \ -(2M+1)^{2d} \ \rho^{-1} \ \ln\,(\epsilon\,(18\,m\,(2M+1)^{2d})^{-1}) \ > \ 0$$

we obtain the conditional probability

$$\begin{aligned} & P\,(\sigma_K < \sqrt{K}\,|\,J \leq m \text{ and } \tau_{i+1} - \tau_i > 2\rho \text{ for all } i = 0, 1, \ldots, J-1) \\ & \qquad \leq \ m\ P\,(\text{the arrow at time } \tau_i \text{ is not good}\,|\,\tau_{i+1} - \tau_i > 2\rho) \\ & \qquad \leq \ m\,(2M+1)^{2d}\,\exp(a_{11}\,(2M+1)^{-2d}\,\rho) \ \leq \ \epsilon/18 \end{aligned} \tag{45}$$

for all $a_{11} < -m_2$. The result follows by observing that the probability to be estimated is smaller than the sum of the three probabilities in (43)–(45). $\square$

With Lemma 22 in hands, we are now ready to prove that the conditional probability given that the initial configuration is sparse that an inner path lasts more than $2\sqrt{K}$ units of time is small when the scaling parameter K is large and the payoff a_{11} is small.

Lemma 23 – *Let $\epsilon > 0$ and $a_{12} < 0$. Then*

$$P\,(\Omega_{in}\,|\,\eta_0 \cap B_{4k} \in \mathcal{S}) \ \leq \ \epsilon/3 \quad \textit{for all } K \textit{ large and } a_{11} < -m_2.$$

PROOF. Recall that $\rho_j := (1/2)(\tau_j + \tau_{j+1})$ and introduce the set-valued process

$$Q_t \ := \ \{y \in B_{4K} : \text{there is an inner path } (x,0) \rightsquigarrow (y,t) \text{ for some } x \in B_{4K}\}.$$

As pointed out above, if $\eta_0 \cap B_{4K}$ is sparse and $\sigma_K > 2\sqrt{K}$ then

- $Q_t \subset \eta_t \cap B_{4K}$ is sparse for all $t \in (\rho_j, \tau_{j+1})$, $j = 0, 1, \ldots, J-1$, and
- card $Q_{\rho_j} \leq$ card Q_{τ_j} for all $j = 1, 2, \ldots, J$.

Since in addition type 1 players with only type 2 neighbors (which is the case for all type 1 players in sparse configurations) change their strategy at rate $-a_{12}$,

$$\lim_{h \to 0} \ h^{-1}\ P\,(Q_{t+h} = Q_t - \{y\}) \ = \ -a_{12} \quad \text{for all } t \in (\rho_j, \tau_{j+1}) \text{ and } y \in Q_t.$$

Using also that Q_t is sparse for at least $\sqrt{K}$ time units before time $2\sqrt{K}$, we obtain

$$\begin{aligned} & P\,(\Omega_{in}\,|\,\eta_0 \cap B_{4K} \in \mathcal{S} \text{ and } \sigma_K > 2\sqrt{K}) \\ & \qquad = \ P\,(Q_t \neq \varnothing \text{ for all } t \in (0, 2\sqrt{K})\,|\,\eta_0 \cap B_{4K} \in \mathcal{S} \text{ and } \sigma_K > 2\sqrt{K}) \\ & \qquad \leq \ (\text{card}\,B_{4K})\ P\,(\text{Exponential}\,(-a_{12}) > \sqrt{K}) \\ & \qquad = \ (\text{card}\,B_{4K})\ \exp(a_{12}\,\sqrt{K}) \ \leq \ \epsilon/6 \end{aligned}$$

for all K large enough. In particular, for all such K and all $a_{11} < -m_2(\epsilon, K, \bar a_{11})$,

$$\begin{aligned} P(\Omega_{in} \,|\, \eta_0 \cap B_{4K} \in \mathcal{S}) \ &\leq \ P(\Omega_{in} \,|\, \eta_0 \cap B_{4K} \in \mathcal{S} \text{ and } \sigma_K > 2\sqrt{K}) \\ &\quad + P(\sigma_K < 2\sqrt{K}) \ \leq \ \epsilon/6 + \epsilon/6 \ = \ \epsilon/3 \end{aligned}$$

according to Lemma 21. This completes the proof. $\square$

The next lemma shows the analog for transversal paths: with probability close to one, none of the transversal paths reaches the target region B_{2K} by time $2\sqrt{K}$.

Lemma 24 – *Let $\epsilon > 0$ and $a_{12} < 0$. For all K large, $P(\Omega_{tr}) \leq \epsilon/3$.*

Proof. We introduce the rates

$$\begin{aligned} \mu_K \ &:= \ -2\sqrt{K}\,(a_{21} + a_{22}) \ \mathrm{card}\,(B_{4K} \setminus B_{4K-M}) \ > \ 0 \\ \nu_K \ &:= \ -2\sqrt{K}\,(a_{21} + a_{22}) \ > \ 0. \end{aligned}$$

First, we observe that

$$\begin{aligned} m_K \ := \ & \text{the number of } (x, 2, j)\text{-arrows} \\ & \text{that point at the region } (B_{4K} \setminus B_{4K-M}) \times (0, 2\sqrt{K}) \ = \ \mathrm{Poisson}\,(\mu_K) \end{aligned}$$

from which it follows that

$$P(m_K > 2\mu_K) \ \leq \ \epsilon/6 \quad \text{for all } K \text{ sufficiently large.} \tag{46}$$

Now, let n_l be the number of type 1 invasion paths of length l

$$(x, t) \rightsquigarrow (y, 2\sqrt{K}) \quad \text{starting from some} \quad (x, t) \in (B_{4K} \setminus B_{4K-M}) \times (0, 2\sqrt{K}), \tag{47}$$

and observe that, on the event that $m_K \leq 2\mu_K$, we have

$$n_l \ \leq \ 2\mu_K\,(2M+1)^{ld} \ = \ -4\sqrt{K}\,(a_{21} + a_{22})(2M+1)^{ld} \ \mathrm{card}\,(B_{4K} \setminus B_{4K-M}). \tag{48}$$

In addition, if in (47) site $y \in B_{2K}$ then the length must be at least

$$l \ \geq \ (2K - M)/M \ \geq \ K/M.$$

Also, since each type 2 player changes her strategy at rate at most $-(a_{21} + a_{22})$, the probability of any given type 1 invasion path (47) of length at least $l \geq K/M$ is bounded by

$$\begin{aligned} P(\mathrm{Poisson}\,(\nu_K) \geq l) \ &= \ \sum_{n=l}^{\infty} \frac{\nu_K^n}{n!}\, e^{-\nu_K} \ \leq \ 2\, \frac{\nu_K^l}{l!}\, e^{-\nu_K} \ \leq \ \frac{4\, e^{-\nu_K}}{\sqrt{2\pi l}} \left(\frac{e\nu_K}{l}\right)^l \\ &\leq \ \left(\frac{e\nu_K}{l}\right)^l \leq \ \left(\frac{eM\nu_K}{K}\right)^l \ = \ (-2\,(a_{21} + a_{22})\, eM/\sqrt{K})^l \end{aligned} \tag{49}$$

for all K sufficiently large where the second inequality follows from Stirling formula. To complete the proof of the lemma, we combine (46), (48) and (49) to obtain

$$\begin{aligned} P(\Omega_{tr}) \ &\leq \ P(m_K > 2\mu_K) \ + \ P(\Omega_{tr} \,|\, m_K \leq 2\mu_K) \\ &\leq \ \epsilon/6 \ + \sum_{l=K/M}^{\infty} 2\mu_K \left[(2M+1)^d\, (-2\,(a_{21} + a_{22})\, eM/\sqrt{K})\right]^l \ \leq \ \epsilon/3 \end{aligned}$$

for all K sufficiently large. $\square$

Having proved Lemmas 21–24, we are now ready to compare the process with oriented site percolation and deduce almost sure extinction of strategy 1. The final part of the proof follows from the same arguments as for Lemma 20. We say that a site $(z, n) \in H$ is **void** whenever

$$\eta_{2n\sqrt{K}} \cap (Kz + B_K) \ = \ \varnothing$$

and define the set $\mathbb{X}_n$ of void sites at level n by

$$\mathbb{X}_n \ := \ \{z \in \mathbb{Z}^d : (z, n) \in H \text{ and is void}\}.$$

The coupling with oriented site percolation is given in the next lemma.

Lemma 25 – *Let $\epsilon > 0$ and $a_{12} < 0$. Then, there exist K large and a coupling of the spatial game with four dependent oriented site percolation such that*

$$\mathbb{W}_n^{1-\epsilon} \subset \mathbb{X}_n \text{ for all } n \quad \text{whenever} \quad \mathbb{X}_0 = \mathbb{W}_0^{1-\epsilon} \text{ and } a_{11} < -\max(m_1, m_2).$$

PROOF. Combining Lemmas 21, 23 and 24, we obtain

$$\begin{array}{rl} P\,(\eta_{2\sqrt{K}} \cap B_{2K} \neq \varnothing \,|\, \eta_0 \cap B_K = \varnothing) & \leq \ P\,(\Omega_{in} \,|\, \eta_0 \cap B_K = \varnothing) \ + \ P\,(\Omega_{tr} \,|\, \eta_0 \cap B_K = \varnothing) \\ & \leq \ P\,(\eta_s \cap B_{4K} \notin \mathcal{S} \text{ for all } s \in (0,1) \,|\, \eta_0 \cap B_K = \varnothing) \\ & \qquad + \ P\,(\Omega_{in} \,|\, \eta_s \cap B_{4K} \in \mathcal{S} \text{ for some } s \in (0,1)) \ + \ P\,(\Omega_{tr} \,|\, \eta_0 \cap B_K = \varnothing) \\ & \leq \ \epsilon/3 + \epsilon/3 + \epsilon/3 \ = \ \epsilon \end{array}$$

for all K large and all $a_{11} < -\max(m_1, m_2)$. The existence of a coupling with oriented site percolation then follows from the same arguments as in the proof of Lemma 16. Note that the comparison can only be made with four dependent percolation because all the events introduced in the proofs of Lemmas 21–24 are measurable with respect to the graphical representation in B_{4K}. $\square$

Repeating the same steps as in the previous two sections, we deduce from the lemma that strategy 2 survives. To also prove extinction of strategy 1, we observe that Lemma 24 excludes the existence of transversal paths that ever intersect B_{2K} by time $2\sqrt{K}$ with probability close to one. In particular, including this event in our definition of void sites, Lemma 25 still holds for arbitrarily small ϵ. Moreover, with this new definition, we obtain that

$$\eta_{2n\sqrt{K}} \cap (Kz + B_K) \ \neq \ \varnothing \quad \text{implies} \quad (w, 0) \to_2 (z, n) \text{ for some } w \in \mathbb{Z}^d$$

in the oriented graph $\mathcal{H}_2$. Note that this implication is similar to the one in the proof of Lemma 20 and can be shown using the same idea. Almost sure extinction of strategy 1 then follows from the lack of percolation of the dry sites repeating again the steps in the proof of Lemma 20.

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